# CHAPTER 6

# *Mechanics of Limit Equilibrium Procedures*

Once appropriate shear strength properties, pore water pressures, slope geometry and other soil and slope properties are established, slope stability calculations need to be performed to ensure that the resisting forces are sufficiently greater than the forces tending to cause a slope to fail. Calculations usually consist of computing a factor of safety using one of several limit equilibrium procedures of analysis. All of these procedures of analysis employ the same definition of the factor of safety and compute the factor of safety using the equations of static equilibrium.

## DEFINITION OF THE FACTOR OF SAFETY

The *factor of safety, F,* is defined with respect to the shear strength of the soil as

$$F = \frac{s}{\tau} \tag{6.1}$$

where $s$ is the available shear strength and $\tau$ is the equilibrium shear stress. The equilibrium shear stress is the shear stress required to maintain a just-stable slope and from Eq. (6.1) may be expressed as

$$\tau = \frac{s}{F} \tag{6.2}$$

The equilibrium shear stress is equal to the available shear strength divided (*factored*) by the factor of safety. The factor of safety represents the factor by which the shear strength must be reduced so that the reduced strength is just in equilibrium with the shear stress ($\tau$) (i.e., the slope is in a state of just-stable *limiting equilibrium.* The procedures used to perform such computations are known as *limit equilibrium procedures.*

The shear strength can be expressed by the Mohr–Coulomb equation. If the shear strength is expressed in terms of total stresses, Eq. (6.2) is written as

$$\tau = \frac{c + \sigma \tan \phi}{F} \tag{6.3}$$

or

$$\tau = \frac{c}{F} + \frac{\sigma \tan \phi}{F} \tag{6.4}$$

where $c$ and $\phi$ are the cohesion and friction angle for the soil, respectively, and $\sigma$ is the total normal stress on the shear plane. The same values for the factor of safety are applied to cohesion and friction in this equation. Equation (6.4) can also be written as

$$\tau = c_d + \sigma \tan \phi_d \tag{6.5}$$

where

$$c_d = \frac{c}{F} \tag{6.6}$$

$$\tan \phi_d = \frac{\tan \phi}{F} \tag{6.7}$$

The quantities $c_d$ and $\phi_d$ represent the developed (or *mobilized*) cohesion and friction angle, respectively.

If the shear strength is expressed in terms of effective stresses (e.g., drained shear strengths are being used), the only change from the above is that Eq. (6.3) is written in terms of effective stresses as



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$$\tau = \frac{c' + (\sigma - u) \tan \phi'}{F} \tag{6.8}$$

where $c'$ and $\phi'$ represent the shear strength parameters in terms of effective stresses, and $u$ is the pore water pressure.

To calculate the factor of safety, a slip surface is assumed and one or more equations of static equilibrium are used to calculate the stresses and factor of safety for each surface assumed. The term *slip surface* is used here to refer to an assumed surface along which sliding or rupture might occur. However, it is the intent of slope stability calculations that sliding and rupture not occur along such surfaces if the slope is designed adequately.

The factor of safety is assumed to be the same at all points along the slip surface. Thus, the value represents an average or overall value for the assumed slip surface. If failure were to occur, the shear stress would be equal to the shear strength at all points along the failure surface and the assumption that the factor of safety is constant would be valid. If, instead, the slope is stable, the factor of safety probably varies along the slip surface (e.g., Wright et al., 1973). However, this should not be of significant consequence as long as the overall factor of safety is suitably greater than 1 and the assumed shear strengths can be fully mobilized along the entire slip surface.

A number of slip surfaces must be assumed to find the slip surface that produces a minimum factor of safety. The surface with the minimum factor of safety is termed the *critical slip surface.* Such a critical surface and the corresponding minimum factor of safety represent the most likely sliding surface, presuming that all of the shear strengths have been determined in a comparable way and with comparable degrees of certainty. Although the slip surface with the minimum factor of safety may not represent a failure mechanism with a significant consequence, the minimum factor of safety is unique for a given problem and should be calculated as part of any analysis of stability. Other slip surfaces with higher factors of safety than the minimum may also be of interest and are discussed in Chapter 13.

**Recapitulation**

- The factor of safety is defined with respect to shear strength.
- The same factor of safety is applied to both cohesion ($c$, $c'$) and friction (tan $\phi$, tan $\phi'$).
- The factor of safety is computed for an assumed slip surface.
- The factor of safety is assumed to be constant along the slip surface.
- A number of different slip surfaces must be assumed and the factor of safety computed for each to determine a critical slip surface with a minimum factor of safety.

## EQUILIBRIUM CONDITIONS

Two different approaches are used to satisfy static equilibrium in the limit equilibrium analysis procedures. Some procedures consider equilibrium for the entire mass of soil bounded beneath by an assumed slip surface and above by the surface of the slope. In these procedures, equilibrium equations are written and solved for a single free body. The infinite slope procedure and the Swedish slip circle method are examples of such single-free-body procedures. In other procedures the soil mass is divided into a number of vertical slices and equilibrium equations are written and solved for each slice. These procedures, termed *procedures of slices,* include such methods as the Ordinary Method of Slices, the Simplified Bishop procedure, and Spencer's Procedure.

Three static equilibrium conditions are to be satisfied: (1) equilibrium of forces in the vertical direction, (2) equilibrium of forces in the horizontal direction, and (3) equilibrium of moments about any point. The limit equilibrium procedures all use at least some static equilibrium equations to compute the factor of safety. Some procedures use and satisfy all of the equilibrium equations, others use and satisfy only some. The Ordinary Method of Slices and Simplified Bishop procedure satisfy only some of the equilibrium requirements. In contrast, Spencer's procedure and the Morgenstern and Price procedure satisfy all the requirements for static equilibrium.

Regardless of whether equilibrium is considered for a single free body or a series of individual vertical slices making up the total free body, there are more unknowns (forces, locations of forces, factor of safety, etc.) than the number of equilibrium equations; the problem of computing a factor of safety is statically indeterminate. Therefore, assumptions must be made to achieve a balance of equations and unknowns. Different procedures make different assumptions to satisfy static equilibrium. Two procedures may even satisfy the same equilibrium conditions but make different assumptions and therefore produce different values for the factor of safety.

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A number of limit equilibrium procedures are described in more detail in the following sections. Each procedure differs from the others in one or more of the ways described above. The different procedures may or may not divide the soil mass into slices, may satisfy different equilibrium conditions, and/or may make different assumptions to obtain a statically determinate solution. The procedures discussed in this chapter were selected because each has a particular advantage or usefulness depending on the slope geometry, the soil strength, and the purpose of the analysis.

**Recapitulation**

- Equilibrium may be considered either for a single free body or for individual vertical slices.
- Depending on the analysis procedure, complete static equilibrium may or may not be satisfied.
- Some assumptions must be made to obtain a statically determinate solution for the factor of safety.
- Different procedures make different assumptions, even when they may satisfy the same equilibrium equations.

## SINGLE FREE-BODY PROCEDURES

The infinite slope, logarithmic spiral, and Swedish slip circle methods all consider equilibrium for a single free-body. These procedures are relatively simple to use and useful within their range of applicability.

### Infinite Slope Procedure

As implied by its name, in the *infinite slope procedure* the slope is assumed to extend infinitely in all directions and sliding is assumed to occur along a plane parallel to the face of the slope (Taylor, 1948). Because the slope is infinite, the stresses will be the same on any two planes that are perpendicular to the slope, such as the planes A–A′ and B–B′ in Figure 6.1. Equilibrium equations are derived by considering a rectangular block like the one shown in Figire 6.1. For an infinite slope, the forces on the two ends of the block will be identical in magnitude, opposite in direction, and collinear. Thus, the forces on the ends of the block exactly balance each other and can be ignored in the equilibrium equations. Summing forces in directions perpendicular and parallel to the slip plane gives the following expressions for the shear force, $S$, and normal force, $N$, on the plane:

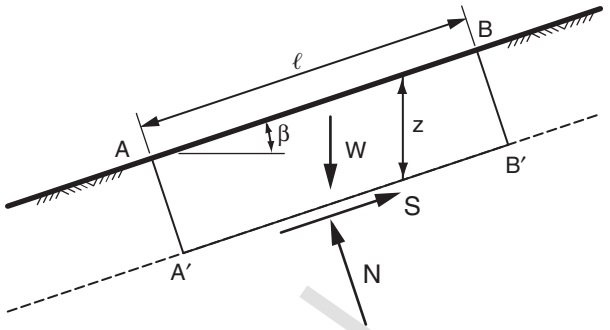



**Figure 6.1** Infinite slope and plane slip surface.

$$S = W \sin \beta \tag{6.9}$$

$$N = W \cos \beta \tag{6.10}$$

where $\beta$ is the angle of inclination of the slope and slip plane, measured from the horizontal, and $W$ is the weight of the block. For a block of unit thickness in the direction perpendicular to the plane of the cross section in Fig. 6.1, the weight is expressed as

$$W = \gamma l z \cos \beta \tag{6.11}$$

where $\gamma$ is the total unit weight of the soil, $l$ the distance between the two ends of the block, measured parallel to the slope, and $z$ the vertical depth to the shear plane. Substituting Eq. (6.11) into Eqs. (6.9) and (6.10) gives

$$S = \gamma l z \cos \beta \sin \beta \tag{6.12}$$

$$N = \gamma l z \cos^2 \beta \tag{6.13}$$

The shear and normal stresses on the shear plane are constant for an infinite slope and are obtained by dividing Eqs. (6.12) and (6.13) by the area of the plane ($l \cdot 1$), to give

$$\tau = \gamma z \cos \beta \sin \beta \tag{6.14}$$

$$\sigma = \gamma z \cos^2 \beta \tag{6.15}$$

Substituting these expressions for the stresses into Eq. (6.3) for the factor of safety for total stresses gives

$$F = \frac{c + \gamma z \cos^2 \beta \tan \phi}{\gamma z \cos \beta \sin \beta} \tag{6.16}$$

For effective stresses, the equation for the factor of safety becomes

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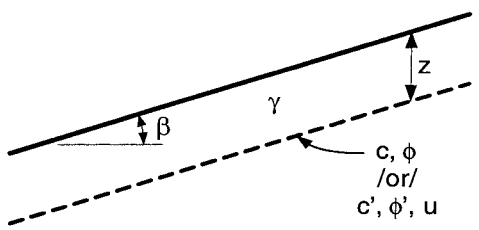
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$$F = \frac{c' + (\gamma z \cos^2\beta - u)\tan \phi'}{\gamma z \cos \beta \sin \beta} \qquad (6.17)$$

Equations for computing the factor of safety for an infinite slope are summarized in Figure 6.2 for both total stress and effective stress analyses and a variety of water and seepage conditions.

For a cohesionless ($c$, $c' = 0$) soil, the factor of safety calculated by an infinite slope analysis is independent of the depth, $z$, of the slip surface. For total stresses (or effective stresses with zero pore water pressure) the equation for the factor of safety becomes

$$F = \frac{\tan \phi}{\tan \beta} \qquad (6.18)$$




Total Stresses: $s = c + \sigma \tan\phi$

Subaerial (not-submerged) slopes:

$$F = \frac{c}{\gamma z}\frac{2}{\sin(2\beta)} + (\cot\beta)\tan\phi$$

Submerged slopes ($\phi = 0$ only):

$$F = \frac{c}{(\gamma - \gamma_w) z \sin(2\beta)}$$

Total Stresses: $s = c' + \sigma' \tan\phi'$

General case (subaerial slope):

$$F = \frac{c'}{\gamma z}\frac{2}{\sin(2\beta)} + \left[\cot\beta - \frac{u}{\gamma z}(\cot\beta + \tan\beta)\right]\tan\phi'$$

Submerged slopes – no flow:

$$F = \frac{c'}{(\gamma - \gamma_w) z \sin(2\beta)}\frac{2}{\sin(2\beta)} + [\cot\beta]\tan\phi'$$

Subaerial slope – seepage parallel to slope face:

$$F = \frac{c'}{\gamma z}\frac{2}{\sin(2\beta)} + \left[\cot\beta - \frac{\gamma_w}{\gamma}(\cot\beta)\right]\tan\phi'$$

Subaerial slope – horizontal seepage:

$$F = \frac{c'}{\gamma z}\frac{2}{\sin(2\beta)} + \left[\cot\beta - \frac{\gamma_w}{\gamma}(\cot\beta + \tan\beta)\right]\tan\phi'$$

Subaerial slope – pore water pressures defined by $r_u = \frac{u}{\gamma z}$:

$$F = \frac{c'}{\gamma z}\frac{2}{\sin(2\beta)} + \left[\cot\beta - r_u(\cot\beta + \tan\beta)\right]\tan\phi'$$

**Figure 6.2** Summary of equations for computing the factor of safety for an infinite slope using both total stresses and effective stresses.

Similarly for effective stresses, if the pore water pressures are proportional to the depth of slide, the factor of safety is expressed by

$$F = [\cot \beta - r_u(\cot \beta + \tan \beta)] \tan \phi' \qquad (6.19)$$

where $r_u$ is the pore water pressure coefficient suggested by Bishop and Morgenstern (1960). The pore water pressure coefficient is defined as

$$r_u = \frac{u}{\gamma z} \qquad (6.20)$$

Because the factor of safety for a cohesionless slope is independent of the depth of the slip surface, it is possible for a slip surface that is only infinitesimally deep to have the same factor of safety as that for deeper surfaces. Regardless of the lateral extent of the slope, a slip surface can develop that is shallow with respect to the lateral dimensions of the slope. Any slope will constitute an infinite slope as long as the soil is cohesionless. Therefore, the infinite slope analysis procedure is the appropriate procedure to use for any slope in cohesionless soil.[1]

The infinite slope analysis is also applicable to slopes in cohesive soils provided that a firmer stratum parallel to the face of the slope limits the depth of the failure surface. If such a stratum exists at a depth that is small compared to the lateral extent of the slope, an infinite slope analysis provides a suitable approximation for stability calculations.

The infinite slope equations were derived by considering equilibrium of forces in two mutually perpendicular directions and thus satisfy all force equilibrium requirements. Moment equilibrium was not considered explicitly; however, the forces on the two ends of the block are collinear and the normal force acts at the center of the block. Thus, moment equilibrium is satisfied, and the infinite slope procedure can be considered to fully satisfy all the requirements for static equilibrium.

[1] An exception to this may occur for soils with curved Mohr failure envelopes that pass through the origin. Although there is no strength at zero normal stress, and thus the soil might be termed *cohesionless,* the factor of safety depends on the depth of slide and the infinite slope analysis may not be appropriate. Also see the example of the Oroville Dam presented in Chapter 7.

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**Recapitulation**

- For a cohesionless slope the factor of safety is independent of the depth of the slip surface, and thus an infinite slope analysis is appropriate (exceptions may occur for nonhomogeneous slopes and/or curved Mohr failure envelopes).
- For cohesive soils the infinite slope analysis procedure may provide a suitable approximation provided that the slip surface is parallel to the slope and limited to a depth that is small compared to the lateral dimensions of the slope.
- The infinite slope analysis procedure fully satisfies static equilibrium.

### Logarithmic Spiral Procedure

In the Logarithmic Spiral procedure, the slip surface is assumed to be a logarithmic spiral, as shown in Figure 6.3 (Frohlich, 1953). A center point and an initial radius, $r_0$, define the spiral. The radius of the spiral varies with the angle of rotation, $\theta$, about the center of the spiral according to the expression

$$r = r_0 e^{\theta \tan \phi_d} \tag{6.21}$$

where $\phi_d$ is the developed friction angle; $\phi_d$ depends on the friction angle of the soil and the factor of safety. The stresses along the slip surface consist of the normal stress ($\sigma$) and the shear stress ($\tau$). The shear stress can be expressed in the case of total stresses by the normal stress, the shear strength parameters ($c$ and $\phi$), and the factor of safety. From Eq. (6.4),

$$\tau = \frac{c}{F} + \sigma \frac{\tan \phi}{F} \tag{6.22}$$

or in terms of developed shear strengths,

$$\tau = c_d + \sigma \tan \phi_d \tag{6.23}$$

A log spiral has the properties that the radius extended from the center of the spiral to a point on the slip surface intersects the slip surface at an angle, $\phi_d$, to the normal (Figure 6.3). Because of this property, the resultant forces produced by the normal stress ($\sigma$) and the frictional portion of the shear stress ($\sigma \tan \phi_d$) act along a line through the center of the spiral and produce no net moment about the center of the spiral. The only forces on the slip surface that produce a moment about the center of the spiral are those due to the developed cohesion. An equilibrium equation may be

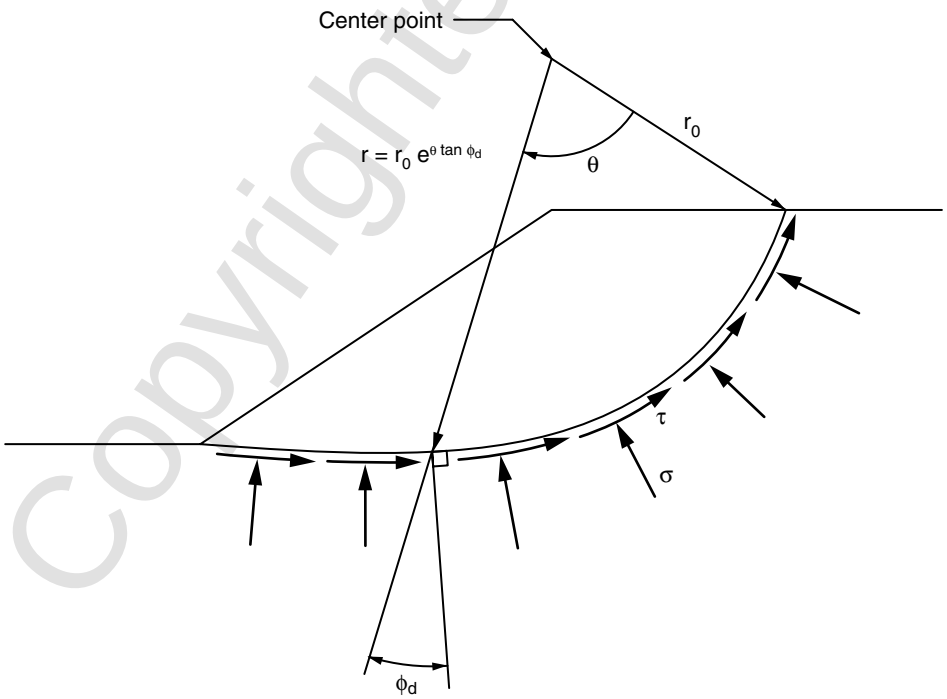



**Figure 6.3** Slope and logarithmic spiral slip surface. (After Frohlich, 1993.)

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written by summing moments about the center of the spiral, which involves only the factor of safety as the unknown. This equation may be used to compute the factor of safety.

In the logarithmic spiral procedure a statically determinant solution is achieved by assuming a particular shape (logarithmic spiral) for the slip surface. By assuming a logarithmic spiral, no additional assumptions are required. Force equilibrium is not considered explicitly in the logarithmic spiral procedure. However, there are an infinite number of combinations of normal and shear stresses along the slip surface that will satisfy force equilibrium. All of these combinations of shear and normal stress will yield the same value for the factor of safety that satisfies moment equilibrium. Thus, the logarithmic spiral implicitly satisfies complete static equilibrium. The logarithmic spiral and infinite slope procedures are the only two limit equilibrium procedures that satisfy complete equilibrium by assuming a specific slope geometry and shape for the slip surface.

Because the logarithmic spiral procedure fully satisfies static equilibrium, it is relatively accurate. Also, for homogeneous slopes, a logarithmic spiral appears to approximate the shape of the most critical potential sliding surface reasonably well. The logarithmic spiral procedure is theoretically the best limit equilibrium procedure for analyses of homogeneous slopes.

For cohesionless ($c$, $c' = 0$) slopes the critical logarithmic spiral that produces the minimum factor of safety has an infinite radius and the spiral coincides with the face of the slope (Figure 6.4). In this case the logarithmic spiral and infinite slope procedures produce identical values for the minimum factor of safety.

The logarithmic spiral equations are relatively complex and awkward for hand calculations, because of the assumed shape of the slip surface. However, the logarithmic spiral procedure is computationally efficient and well suited for implementation in computer calculations. The procedure is useful for performing the computations required to produce slope stability charts, and once such charts have been developed there is little need for using the detailed logarithmic spiral equations (Wright, 1969; Leshchinsky and Volk, 1985; Leschinsky and San, 1994). The logarithmic spiral procedure has also received recent interest and attention for use in software to analyze reinforced slopes, particularly for design software that must perform many repetitive calculations to find a suitable arrangement for reinforcement (Leshchinsky, 1997).

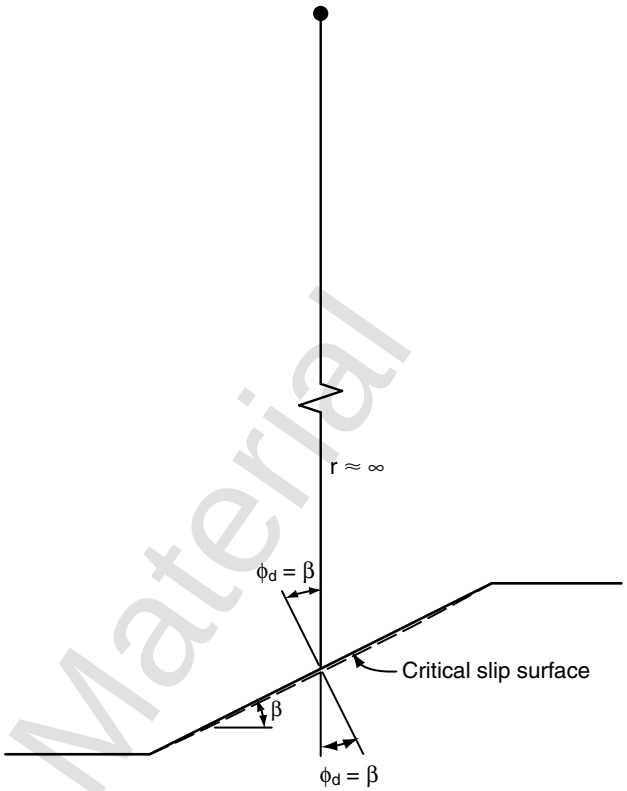



**Figure 6.4** Critical logarithmic spiral slip surface for a cohesionless slope.

**Recapitulation**

- The logarithmic spiral procedure achieves a statically determinate solution by assuming a specific logarithmic spiral shape for the slip surface [Eq. (6.21)].
- The logarithmic spiral procedure explicitly satisfies moment equilibrium and implicitly satisfies complete force equilibrium. Because complete equilibrium is satisfied, the procedure is relatively accurate.
- The logarithmic spiral procedure is theoretically the best procedure for analysis for homogeneous slopes. Much of the effort required can be reduced by use of dimensionless slope stability charts (Leshchinsky, 1985, 1994).
- The logarithmic spiral procedure is used in several computer programs for design of reinforced slopes using geogrids, soil nails, and so on.

### Swedish Circle/$\phi = 0$ Method

In the *Swedish Circle method* the slip surface is assumed to be a circular arc and moments are summed

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about the center of the circle to calculate a factor of safety. Some form of the method was apparently first used by Petterson in about 1916 (Petterson, 1955), but the method seems to have first been formalized for $\phi = 0$ by Fellenius in 1922 (Fellenius, 1922; Skempton, 1948). The friction angle is assumed to be zero and thus the shear strength is assumed to be due to ''cohesion'' only. For this reason, the Swedish Circle method is also called the $\phi = 0$ *method.*

The Swedish Circle or ''$\phi = 0$'' method is actually a special case of the logarithmic spiral procedure: When $\phi = 0$, a logarithmic spiral becomes a circle. However, the equilibrium equations for a circle are much simpler than those for a more general logarithmic spiral, and thus the Swedish Circle method is generally considered to be a separate method. The Swedish Circle method also seems to have preceded the development of the logarithmic spiral method for slope stability analysis.

Referring to the slope and circular slip surface shown in Figure 6.5, the driving (overturning) moment tending to produce rotation of the soil about the center of the circle is given by

$$M_d = Wa \tag{6.24}$$

where $W$ is the weight of the soil mass above the circular slip surface and $a$ is the horizontal distance between the center of the circle and the center of gravity of the soil mass; $a$ is the moment arm. The resisting moment is provided by the shear stresses ($\tau$) acting along the circular arc. For a unit thickness of the cross section shown in Figure 6.5, the resisting moment is given by

$$M_r = \tau l r \tag{6.25}$$

where $l$ is the length of the circular arc and $r$ is the radius. For equilibrium, the resisting and overturning moments must balance. Thus,

$$Wa = \tau l r \tag{6.26}$$

The shear stress in this equation can be expressed in terms of the shear strength and factor of safety using Eq. (6.1). Introducing Eq. (6.1) and replacing the shear strength by the cohesion $c$ yields

$$Wa = \frac{clr}{F} \tag{6.27}$$

or, after rearranging,

$$F = \frac{clr}{Wa} \tag{6.28}$$

Equation (6.28) is the equation used to compute the factor of safety by the Swedish Circle method.

The term $clr$ in the numerator of Eq. (6.28) represents the available resisting moment; the term $Wa$ in the denominator represents the driving moment. Therefore, the factor of safety in this case is equal to the

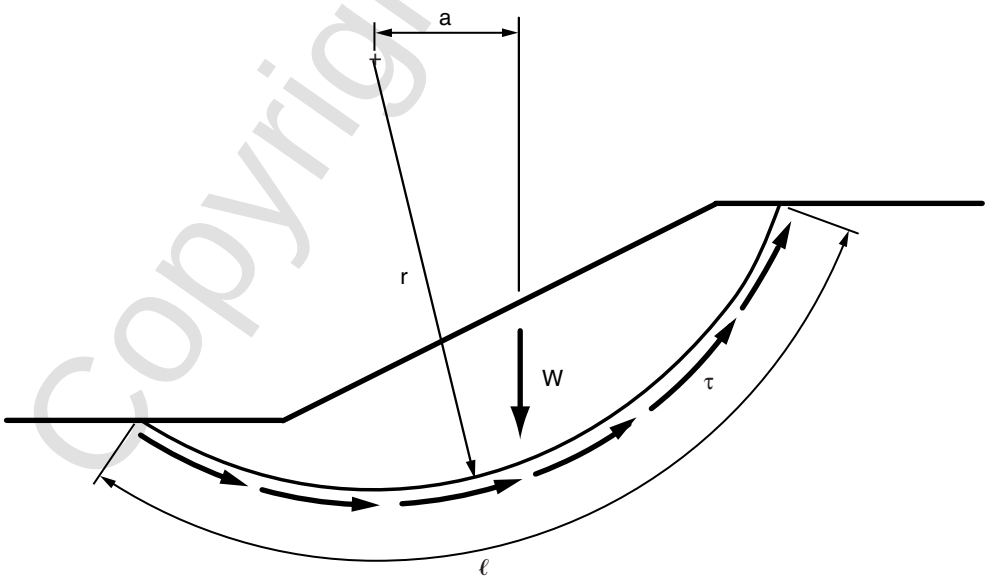



**Figure 6.5** Slope and slip surface for the Swedish circle ($\phi = 0$) procedure.

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available resisting moment, $M_r$, divided by the actual driving moment, $M_d$:

$$F = \frac{\text{available resisting moment}}{\text{actual driving moment}} \tag{6.29}$$

The Swedish Circle method can be derived by starting with Eq. (6.29) as the definition of the factor of safety, and this approach is sometimes used. There are also other definitions that have been suggested for the factor of safety, and these are discussed later under the head "Alternative Definitions for the Factor of Safety." However, the authors prefer to begin with the definition of the factor of safety in terms of shear strength given by Eq. (6.1) rather than Eq. (6.29).

Because the Swedish Circle method is a special case of the logarithmic spiral method, it also satisfies complete static equilibrium. Both the logarithmic spiral and $\phi = 0$ methods use only the equilibrium equation for summation of moments about the center point of the slip surface, but all conditions of static equilibrium are implicitly satisfied. The $\phi = 0$ method achieves a statically determinate solution by assuming that $\phi = 0$ and a circular slip surface. No direct assumptions are made about the unknown forces that contribute to equilibrium.

Equation (6.28) was derived for a constant value of cohesion, but the equation is easily extended to cases where the cohesion varies. If $c$ varies, the circular slip surface is subdivided into an appropriate number of segments of length, $\Delta l_i$, each with a corresponding average strength, $c_i$ (Figure 6.6). The expression for the resisting moment becomes

$$M_r = \frac{\sum(c_i \, \Delta l_i r)}{F} \tag{6.30}$$

where the summation is performed for the segments along the slip surface. The equation for the factor of safety is then

$$F = \frac{r \sum(c_i \, \Delta l_i)}{Wa} \tag{6.31}$$

The term $Wa$ in Eqs. (6.28) and (6.31) represents the driving moment due to the weight of the soil. To compute the moment arm, $a$, it is necessary to compute the center of gravity of the soil mass above the slip surface, which is not easy considering the usually odd shape of the soil mass. Instead, in practice, calculations are usually performed using either slope stability charts like those described in the Appendix or one of the procedures of slices that are described in the next section. The procedures of slices provide a more convenient way to calculate the driving moment, and for a circular slip surface the procedures of slices produce the same value for the factor of safety as the Swedish Circle method.

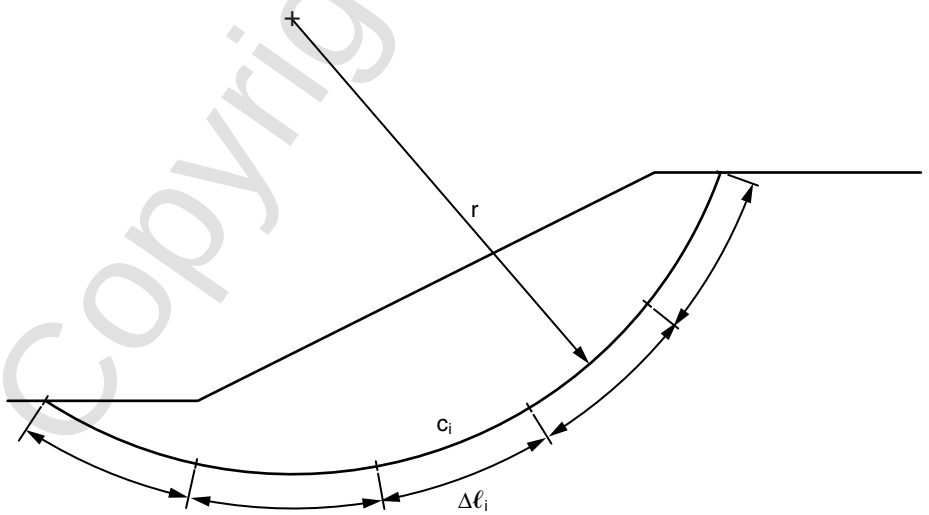



**Figure 6.6** Circular slip surface subdivided into segments when the undrained shear strength varies.

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**Recapitulation**

- The Swedish Circle (or $\phi = 0$) method explicitly satisfies moment equilibrium and implicitly satisfies force equilibrium completely.
- The Swedish Circle (or $\phi = 0$) method is an accurate method of slope stability analysis for both homogeneous and inhomogeneous slopes in $\phi = 0$ soils, provided that the slip surface can be approximated by a circle.

## PROCEDURES OF SLICES: GENERAL

In the remaining procedures covered in this chapter, the soil mass above the slip surface is subdivided into a number of vertical slices. The actual number of slices used depends on the slope geometry and soil profile and is discussed in more detail in Chapter 14.

Some procedures of slices assume a circular slip surface while others assume an arbitrary (noncircular) slip surface. Procedures that assume a circular slip surface consider equilibrium of moments about the center of the circle for the entire free body composed of all slices. In contrast, the procedures that assume an arbitrary shape for the slip surface usually consider equilibrium in terms of the individual slices. It is appropriate to consider the procedures for circular and noncircular slip surfaces separately.

## PROCEDURES OF SLICES: CIRCULAR SLIP SURFACES

Procedures based on a circular slip surface consider equilibrium of moments about the center of the circle. Referring to the slope and circular slip surface shown in Figure 6.7, the overturning moment can be expressed as

$$M_d = \sum W_i a_i \tag{6.32}$$

where $W_i$ is the weight of the $i$th slice and $a_i$ is the horizontal distance between the center of the circle and the center of the slice. Distances toward the crest of the slope, to the right of the center shown in Figure 6.7, are positive; distances toward the toe of the slope, to the left of the center, are negative. Although theoretically the moment arm, $a_i$, is measured from the center of the circle to the center of gravity of the slice, a sufficient number of slices is generally used that the differences between the center (midwidth) and center of gravity of the slice are negligible. In most cases $a_i$ is measured from the center of the circle to the center (midwidth) of the slice.

The moment arm, $a_i$, in Eq. (6.32) can be expressed in terms of the radius of the circle and the inclination of the bottom of the respective slice. Although the base of the slice is curved, the base can be assumed to be a straight line, as suggested in Figure 6.7, with negligible loss in accuracy. The inclination of the base of

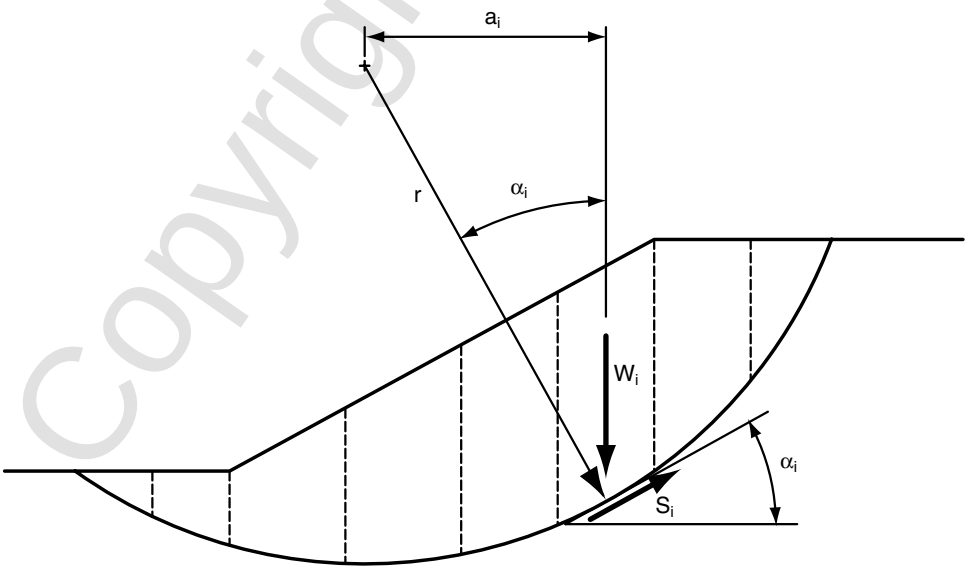



**Figure 6.7** Circular slip surface with overlying soil mass subdivided into vertical slices.

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the slice is represented by the angle, $\alpha_i$, measured between the base of the slice and the horizontal. Positive and negative values are indicated in Figure 6.7. The angle between a line extended from the center of the circle to the center of the base of the slice and a vertical line is also equal to the angle, $\alpha_i$ (Figure 6.7). Thus, the moment arm ($a_i$) is expressed by

$$a_i = r \sin \alpha_i \tag{6.33}$$

and the driving moment becomes

$$M_d = r \sum W_i \sin \alpha_i \tag{6.34}$$

The radius in Eq. (6.34) has been moved outside the summation because the radius is constant for a circle.

The resisting moment is provided by the shear stresses ($\tau$) on the base of each slice; normal stresses ($\sigma$) on the base of each slice act through the center of the circle and thus produce no moment. The total resisting moment for all slices is

$$M_r = \sum rS_i = r \sum S_i \tag{6.35}$$

where $S_i$ is the shear force on the base of the $i$th slice and the summation is performed for all slices. The shear force is the product of the shear stress, $\tau_i$, and the area of the base of the slice, which for a slice of unit thickness is $\Delta l_i \cdot 1$. Thus,

$$M_r = r \sum \tau_i \, \Delta l_i \tag{6.36}$$

The shear stress can be expressed in terms of the shear strength and the factor of safety by Eq. (6.1) to give

$$M_r = r \sum \frac{s_i \, \Delta l_i}{F} \tag{6.37}$$

Equating the resisting moment [Eq. (6.37)] and the driving moment [Eq. (6.34)] and rearranging, the following equation can be written for the factor of safety:

$$F = \frac{\sum s_i \, \Delta l_i}{\sum W_i \sin \alpha_i} \tag{6.38}$$

The radius has been canceled from both the numerator and denominator of this equation. However, the equation is still valid only for a circular slip surface.

At this point the subscript $i$ will be dropped from use with the understanding that the quantities inside the summation are the values for an individual slice and that the summations are performed for all slices. Thus, Eq. (6.38) is written as

$$F = \frac{\sum s \Delta l}{\sum W \sin \alpha} \tag{6.39}$$

For total stresses the shear strength is expressed by

$$s = c + \sigma \tan \phi \tag{6.40}$$

Substituting this into Eq. (6.39), gives

$$F = \frac{\sum (c + \sigma \tan \phi) \, \Delta l}{\sum W \sin \alpha} \tag{6.41}$$

Equation (6.41) represents the static equilibrium equation for moments about the center of a circle. If $\phi$ is equal to zero, Eq. (6.41) becomes

$$F = \frac{\sum c \, \Delta l}{\sum W \sin \alpha} \tag{6.42}$$

which can be solved for a factor of safety. Equations (6.42) and (6.31), derived earlier for the Swedish Circle ($\phi = 0$) method, both satisfy moment equilibrium about the center of a circle and make no assumptions other than that $\phi = 0$ and the slip surface is a circle. Therefore, both equations produce the same value for the factor of safety. The only difference is that Eq. (6.31) considers the entire free body as a single mass; Eq. (6.42) subdivides the mass into slices. Equation (6.42), based on slices, is more convenient to use than Eq. (6.31) because it avoids the need to locate the center of gravity of what may be an odd-shaped soil mass above the slip surface.

If the friction angle is not equal to zero, the equation presented above for the factor of safety [Eq. (6.41)] requires that the normal stress on the base of each slice be known. The problem of determining the normal stress is statically indeterminate and requires that additional assumptions be made in order to compute the factor of safety. The Ordinary Method of Slices and the Simplified Bishop procedures described in the next two sections make two different sets of assumptions to obtain the normal stress on the base of the slices and, subsequently, the factor of safety.

### Ordinary Method of Slices

The Ordinary Method of Slices is a procedure of slices that neglects the forces on the sides of the slice. The Ordinary Method of Slices has also been referred to as

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the ''Swedish Method of Slices'' and the ''Fellenius Method.'' This method should not, however, be confused with the U.S. Army Corps of Engineers' Modified Swedish method, which is described later. Similarly, the method should not be confused with other methods of slices that Fellenius developed, including a method of slices that fully satisfies static equilibrium (Fellenius, 1936).

Referring to the slice shown in Figure 6.8 and resolving forces perpendicular to the base of the slice, the normal force for the Ordinary Method of Slices can be expressed as

$$N = W \cos \alpha \tag{6.43}$$

The normal force expressed by Eq. (6.43) is the same as the normal force that would exist if the resultant force due to the forces on the sides of the slice acted in a direction parallel to the base of the slice (Bishop, 1955). However, it is impossible for this to occur and for the forces on the slice to be in equilibrium unless the interslice forces are zero.

The normal stress on the base of a slice is obtained by dividing the normal force by the area of the base of the slice ($1 \cdot \Delta l$), to give

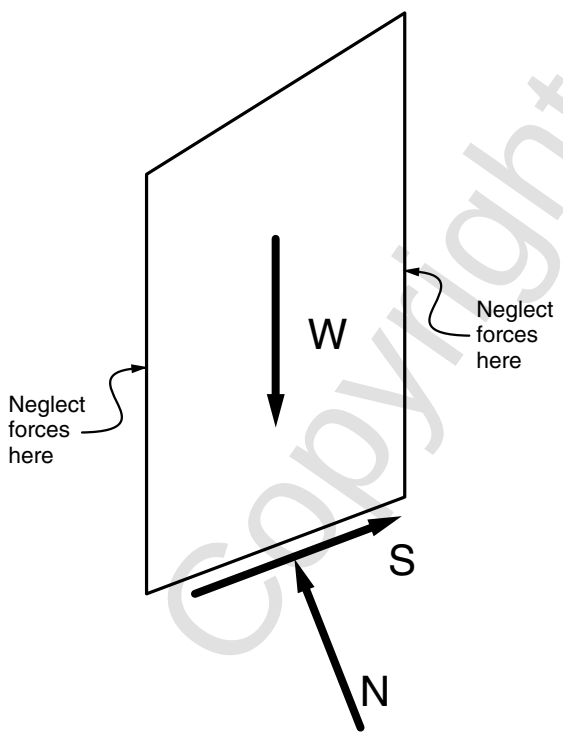



**Figure 6.8** Slice with forces considered in the Ordinary Method of Slices.

$$\sigma = \frac{W \cos \alpha}{\Delta l} \tag{6.44}$$

Substituting this expression for the normal force into Eq. (6.41), derived above for the factor of safety from moment equilibrium, gives the following equation for the factor of safety:

$$F = \frac{\sum (c\ \Delta l + W \cos \alpha \tan \phi)}{\sum W \sin \alpha} \tag{6.45}$$

Equation (6.45) is the equation for the factor of safety by the Ordinary Method of Slices when the shear strength is expressed in terms of total stresses.

When the shear strength is expressed in terms of effective stresses the equation for the factor of safety from moment equilibrium is

$$F = \frac{\sum (c' + \sigma' \tan \phi')\ \Delta l}{\sum W \sin \alpha} \tag{6.46}$$

where $\sigma'$ is the effective normal stress, $\sigma - u$. From Eq. (6.44) for the total normal stress, the effective normal stress can be expressed as

$$\sigma' = \frac{W \cos \alpha}{\Delta l} - u \tag{6.47}$$

where $u$ is the pore water pressure on the slip surface. Substituting this expression for the effective normal stress [Eq. (6.47)] into the equation for the factor of safety (6.46) and rearranging gives

$$F = \frac{\sum [c'\ \Delta l + (W \cos \alpha - u\ \Delta l) \tan \phi']}{\sum W \sin \alpha} \tag{6.48}$$

Equation (6.48) represents an expression for the factor of safety by the Ordinary Method of Slices for effective stresses. However, the assumption involved in this equation ($\sigma = W \cos \alpha / \Delta l$) can lead to unrealistically low, even negative values for the effective stresses on the slip surface. This can be demonstrated as follows: Let the weight of the slice be expressed as

$$W = \gamma h b \tag{6.49}$$

where $h$ is the height of the slice at the centerline and $b$ is the width of the slice (Figure 6.9). The width of the slice is related to the length of the base of the slice, $\Delta l$, as

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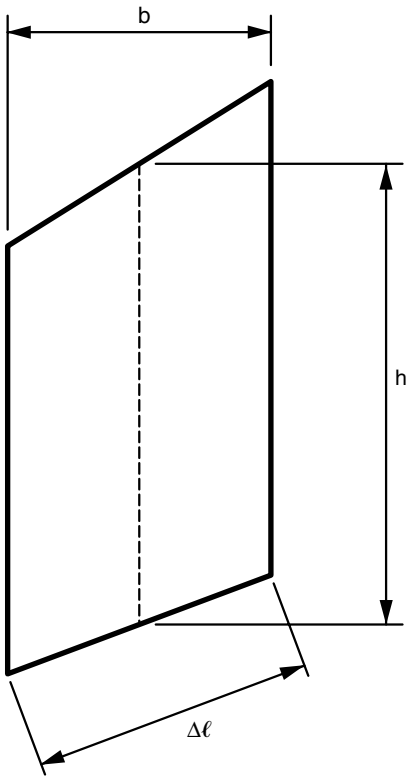



**Figure 6.9** Dimensions for an individual slice.

$$b = \Delta l \cos \alpha \tag{6.50}$$

Thus, Eq. (6.49) can be written as

$$W = \gamma h \, \Delta l \cos \alpha \tag{6.51}$$

Substituting this expression for the weight of the slice into Eq. (6.48) and rearranging gives

$$F = \frac{\sum [c' \Delta l + (\gamma h \cos^2 \alpha - u) \Delta l \tan \phi']}{\sum W \sin \alpha} \tag{6.52}$$

The expression in parentheses, $\gamma h \cos^2 \alpha - u$, represents the effective normal stress, $\sigma'$, on the base of the slice. Therefore, we can also write

$$\frac{\sigma'}{\gamma h} = \cos^2 \alpha - \frac{u}{\gamma h} \tag{6.53}$$

where the ratio $\sigma'/\gamma h$ is the ratio of effective normal stress to total overburden pressure and $u/\gamma h$ is the ratio of pore water pressure to total vertical overburden pressure. Let's now suppose that the pore water pressure is equal to one-third the overburden pressure (i.e., $u/\gamma h = \frac{1}{3}$). Further suppose that the slip surface is inclined upward at an angle, $\alpha$, of 60° from the horizontal. Then, from Eq. (6.53),

$$\frac{\sigma'}{\gamma h} = \cos^2 (60°) - \tfrac{1}{3} = -0.08 \tag{6.54}$$

which indicates that the effective normal stress is negative! Negative values will exist for the effective stress in Eq. (6.52) as the pore water pressures become larger and the slip surface becomes steeper ($\alpha$ becomes large). The negative values occur because the forces on the sides of the slice are ignored in the Ordinary Method of Slices and there is nothing to counteract the pore water pressure.

By first expressing the weight of the slice in terms of an effective weight and then resolving forces perpendicular to the base of the slice, a better expression for the factor of safety can be obtained for the Ordinary Method of Slices (Turnbull and Hvorslev, 1967). The effective slice weight, $W'$, is given by

$$W' = W - ub \tag{6.55}$$

The term $ub$ represents the vertical *uplift force* due to the pore water pressure on the bottom of the slice. The uplift force acts to counterbalance the weight of the slice. Resolving forces due to the effective stresses in a direction perpendicular to the base of the slice gives the effective normal force, $N'$,

$$N' = W' \cos \alpha \tag{6.56}$$

or from Eqs. (6.50) and (6.55),

$$N' = W \cos \alpha - u \, \Delta l \cos^2 \alpha \tag{6.57}$$

The effective normal stress, $\sigma'$, is obtained by dividing this force by the area of the base of the slice, which gives

$$\sigma' = \frac{W \cos \alpha}{\Delta l} - u \cos^2 \alpha \tag{6.58}$$

Finally, introducing Eq. (6.58) for the effective normal stress into Eq. (6.46) for the factor of safety derived from moment equilibrium gives

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$$F = \frac{\sum [c' \, \Delta l + (W \cos \alpha - u \, \Delta l \cos^2 \alpha) \tan \phi']}{\sum W \sin \alpha} \tag{6.59}$$

This alternative expression for the factor of safety by the Ordinary Method of Slices does not result in negative effective stresses on the slip surface as long as the pore water pressures are less than the total vertical overburden pressure, a condition that must clearly exist for any reasonably stable slope.

**Recapitulation**

- The Ordinary Method of Slices assumes a circular slip surface and sums moments about the center of the circle; the method only satisfies moment equilibrium.
- For $\phi = 0$ the Ordinary Method of Slices gives exactly the same value for the factor of safety as does the Swedish Circle method.
- The Ordinary Method of Slices permits the factor of safety to be calculated directly. All of the other procedures of slices described subsequently require an iterative, trial-and-error solution for the factor of safety. Thus, the method is convenient for hand calculations.
- The Ordinary Method of Slices is less accurate than are other procedures of slices. The accuracy is less for effective stress analyses and decreases as the pore water pressures become larger.
- Accuracy of the Ordinary Method of Slices can be improved by using Eq. (6.59) rather than Eq. (6.48) for effective stress analyses.

### Simplified Bishop Procedure

In the Simplified Bishop procedure the forces on the sides of the slice are assumed to be horizontal (i.e., there are no shear stresses between slices). Forces are summed in the vertical direction to satisfy equilibrium in this direction and to obtain an expression for the normal stress on the base of each slice. Referring to the slice shown in Figure 6.10 and resolving forces in the vertical direction, the following equilibrium equation can be written for forces in the vertical direction:

$$N \cos \alpha + S \sin \alpha - W = 0 \tag{6.60}$$

Forces are considered positive when they act upward. The shear force in Eq. (6.60) is related to the shear stress by

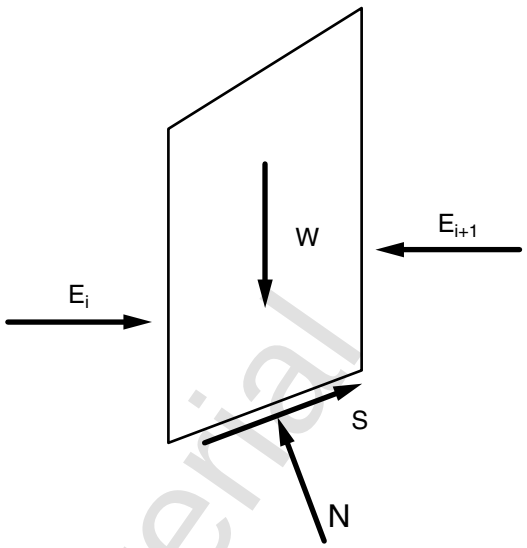



**Figure 6.10** Slice with forces for the Simplified Bishop procedure.

$$S = \tau \, \Delta l \tag{6.61}$$

or in terms of the shear strength and factor of safety [Eq. (6.2)], we can write

$$S = \frac{s \, \Delta l}{F} \tag{6.62}$$

For shear strengths expressed in terms of effective stresses with the Mohr–Coulomb strength equation, we can write

$$S = \frac{1}{F}[c' \, \Delta l + (N - u \, \Delta l) \tan \phi'] \tag{6.63}$$

Combining Eqs. (6.60) and (6.63) and solving for the normal force, $N$, we obtain

$$N = \frac{W - (1/F)(c' \, \Delta l - u \, \Delta l \tan \phi') \sin \alpha}{\cos \alpha + (\sin \alpha \tan \phi')/F} \tag{6.64}$$

the effective normal stress on the base of the slice is given by

$$\sigma' = \frac{N}{\Delta l} - u \tag{6.65}$$

Combining Eqs. (6.64) and (6.65) and introducing them into the equation for equilibrium of moments

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about the center of a circle for effective stresses [Eq. (6.46)], we can write, after rearranging terms,

$$F = \frac{\sum\left[\dfrac{c'\,\Delta l\cos\alpha + (W - u\,\Delta l\cos\alpha)\tan\phi'}{\cos\alpha + (\sin\alpha\tan\phi')/F}\right]}{\sum W\sin\alpha} \tag{6.66}$$

Equation (6.66) represents the equation for the factor of safety for the Simplified Bishop procedure.

Equation (6.66) was derived with the shear strength expressed in terms of effective stresses. The only distinction between total and effective stresses that is made in deriving any equation for the factor of safety is in whether the shear strength is expressed in terms of total stresses or effective stresses [e.g., Eq. (6.3) vs. Eq. (6.8)]. An equation for the factor of safety based on total stresses can be obtained from the equation for effective stresses by replacing the effective stress shear strength parameters ($c'$ and $\phi'$) by their total stress equivalents ($c$ and $\phi$) and setting the pore water pressure term ($u$) to zero. Thus, the equation for the factor of safety in terms of total stresses for the Simplified Bishop procedure is

$$F = \frac{\sum\left[\dfrac{c\,\Delta l\cos\alpha + W\tan\phi}{\cos\alpha + (\sin\alpha\tan\phi)/F}\right]}{\sum W\sin\alpha} \tag{6.67}$$

In many problems, the shear strength will be expressed in terms of total stresses (e.g., UU strengths) for some materials and in terms of effective stresses (e.g., CD strengths) for other materials. Thus, the terms being summed in the numerator of Eq. (6.66) or (6.67) will contain a mixture of effective stresses and total stresses, depending on the applicable drainage conditions along the slip surface (base of each slice).

For saturated soils and undrained loading, the shear strength may be characterized using total stresses with $\phi = 0$. In this case Eq. (6.67) reduces further to

$$F = \frac{\sum c\Delta l}{\sum W\sin\alpha} \tag{6.68}$$

Equation (6.68) is identical to Eq. (6.42) derived for the Ordinary Method of Slices. In this case ($\phi = 0$) the logarithmic spiral, Swedish Circle, Ordinary Method of Slices, and Simplified Bishop procedures all give the same value for the factor of safety. In fact, any procedure that satisfies equilibrium of moments about the center of a circular slip surface will give the same value for the factor of safety for $\phi = 0$ conditions.

Although the Simplified Bishop procedure does not satisfy complete static equilibrium, the procedure gives relatively accurate values for the factor of safety. Bishop (1955) showed that the procedure gives improved results over the Ordinary Method of Slices, especially when analyses are being performed using effective stresses and the pore water pressure are relatively high. Also, good agreement has been shown between the factors of safety calculated by the Simplified Bishop procedure and limit equilibrium procedures that fully satisfy static equilibrium (Bishop, 1955; Fredlund and Krahn, 1977; Duncan and Wright, 1980). Also Wright et al. (1973) have shown that the factor of safety calculated by the Simplified Bishop procedure agrees favorably (within about 5%) with the factor of safety calculated using stresses computed independently using finite element procedures. The primary practical limitation of the Simplified Bishop procedure is that it is restricted to circular slip surfaces.

**Recapitulation**

- The Simplified Bishop procedure assumes a circular slip surface and horizontal forces between slices. Moment equilibrium about the center of the circle and force equilibrium in the vertical direction for each slice are satisfied.
- For $\phi = 0$ the Simplified Bishop procedure gives the same, identical value for the factor of safety as the Swedish Circle and Ordinary Method of Slices procedures because all these procedures satisfy moment equilibrium about the center of a circle and that produces a unique value for the factor of safety.
- The Simplified Bishop procedure is more accurate than the Ordinary Method of Slices, especially for effective stress analyses with high pore water pressures.

### Inclusion of Additional Known Forces

The equations presented above for the factor of safety by both the Ordinary Method of Slices and Simplified Bishop procedures are based on the assumption that the only driving forces are due to the weight of the soil mass, and the only resisting forces are those due to the shear strength of the soil. Frequently, there are additional known driving and resisting forces: Slopes that have water adjacent to them or support additional surcharge loads due to traffic or stockpiled materials are subjected to additional loads. Also, the pseudostatic

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analyses for seismic loading, which are discussed in Chapter 10, involve an additional horizontal body force on the slices to represent earthquake loading. Finally, stability computations for reinforced slopes include additional forces to represent the reinforcement. All these forces are considered to be known forces; that is, they are prescribed as part of the definition of the problem and must be included in the equilibrium equations to compute the factor of safety. However, because the additional forces are known, they can be included in the equilibrium equations without requiring any additional assumptions to achieve a statically determinate solution. The inclusion of additional forces is shown below using the Simplified Bishop procedure for illustration.

Consider first the equation of overall moment equilibrium about the center of a circle. With only forces due to the weight and shear strength of the soil, equilibrium is expressed by

$$r \sum \frac{s_i \, \Delta l_i}{F} - r \sum W_i \sin \alpha_i = 0 \qquad (6.69)$$

where counterclockwise (resisting) moments are considered positive and clockwise (overturning) moments are considered negative. Instead, if there are also seismic forces, $kW_i$, and forces due to soil reinforcement, $T_i$ (Figure 6.11), the equilibrium equation might be written as

$$r \sum \frac{s_i \, \Delta l_i}{F} - r \sum W_i \sin \alpha_i - \sum kW_i d_i + \sum T_i h_i = 0 \qquad (6.70)$$

where $k$ is the seismic coefficient, $d_i$ the vertical distance between the center of the circle and the center of gravity of the slice, $T_i$ represents the force in the reinforcement where the reinforcement crosses the slip surface, and $h_i$ is the moment arm of the reinforcement force about the center of the circle. The summation, $\Sigma\, kW_i d_i$, is performed for all slices, while the summation, $\Sigma\, T_i h_i$, applies only to slices where the reinforcement intersects the slip surface. The reinforcement shown in Figure 6.11 is horizontal, and thus the moment arm is simply the vertical distance between the reinforcement and the center of the circle. This, however, may not always be the case. For example, in Figure 6.12 a slice is shown where the reinforcement force is inclined at an angle, $\psi$, from the horizontal.

Because the forces represented by the last two summations in Eq. (6.70) involve only known quantities, it is convenient to replace these summations by a single term, $M_n$, that represents the net moment due to the known forces. The known forces may include seismic forces, reinforcement forces, and in the case of the slice shown in Figure 6.12, an additional moment due to a force, $P$, on the top of the slice. Equation (6.70) is then written as

$$r \sum \frac{s_i \, \Delta l_i}{F} - r \sum W_i \sin \alpha_i + M_n = 0 \qquad (6.71)$$

Positive values for $M_n$ represent a net counterclockwise moment; negative values represent a net clockwise mo-

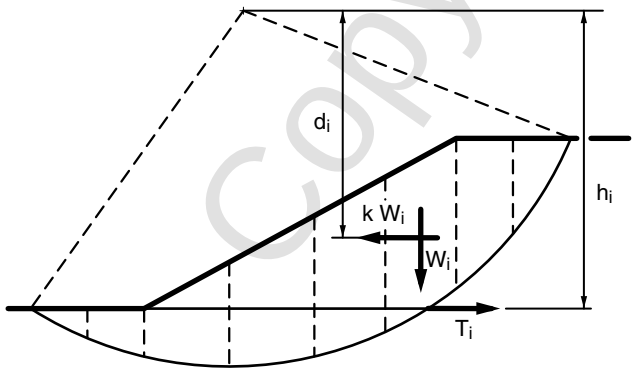



**Figure 6.11** Slope with additional known seismic and reinforcement forces.

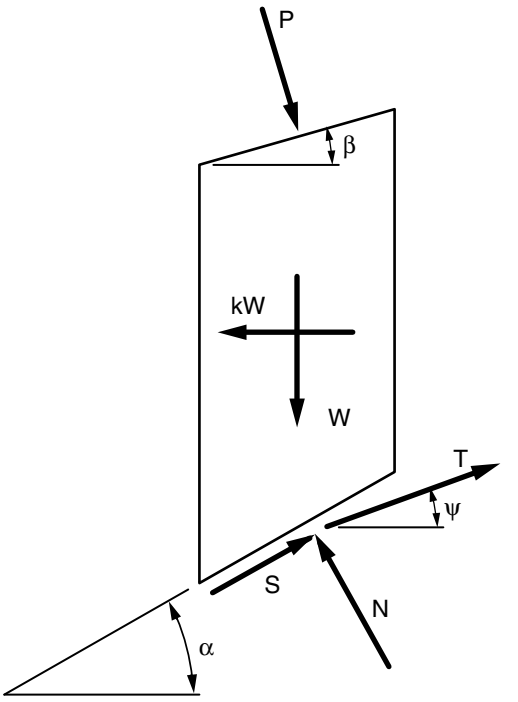



**Figure 6.12** Individual slice with additional known forces.

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ment. The equation for the factor of safety that satisfies moment equilibrium then becomes

$$F = \frac{\sum s_i \, \Delta l_i}{\sum W_i \sin \alpha_i - M_n/r} \tag{6.72}$$

If the shear strength, $s$, is expressed in terms of effective stresses, and the subscripts $i$ are now dropped with the understanding that the terms inside each summation apply to an individual slice, Eq. (6.72) can be written as

$$F = \frac{\sum [c' + (\sigma - u) \tan\phi'] \, \Delta l}{\sum W \sin \alpha - M_n/r} \tag{6.73}$$

To determine the normal stress, $\sigma$ ($= N/\Delta l$) in Eq. (6.73), the equation for equilibrium of forces in the vertical direction is used again. Suppose that the slice contains the known forces shown in Figure 6.12. The known forces in this instance consist of a seismic force, $kW$, a force, $P$, due to water loads on the surface of the slope, and force, $T$, due to reinforcement intersecting the base of the slice. The force $P$ acts perpendicular to the top of the slice, and the reinforcement force is inclined at an angle, $\psi$, from the horizontal. Summation of forces in the vertical direction gives

$$N \cos \alpha + S \sin \alpha - W - P \cos \beta + T \sin \psi = 0 \tag{6.74}$$

where $\beta$ is the inclination of the top of the slice and $\psi$ represents the inclination of the reinforcement from the horizontal. Equation (6.74) is based on the Simplified Bishop assumption that there are no shear forces on the sides of the slice (i.e., the interslice forces are horizontal). Note that because the seismic force is assumed to be horizontal, it does not contribute to equilibrium in the vertical direction; however, if there were a seismic force component in the vertical direction, the vertical component would appear in Eq. (6.74). It is again convenient to combine the contribution of the known forces into a single quantity, represented in this case by a vertical force, $F_v$, which includes the vertical components of all of the known forces except the slice weight,[2] that is,

$$F_v = -P \cos \beta + T \sin \psi \tag{6.75}$$

Positive forces are assumed to act upward; negative forces act downward. The summation of forces in the vertical direction then becomes

$$N \cos \alpha + S \sin \alpha - W + F_v = 0 \tag{6.76}$$

Introducing the Mohr–Coulomb strength equation, which includes the definition of the factor of safety [Eq. (6.63)], into Eq. (6.76) and then solving for the normal force, $N$, gives

$$N = \frac{W - F_v - (1/F)(c' \, \Delta l - u \, \Delta l \tan \phi')\sin \alpha}{\cos \alpha + (\sin \alpha \tan \phi')/F} \tag{6.77}$$

Combining Eq. (6.77) for the normal force with Eq. (6.73) for the factor of safety then gives

$$F = \frac{\sum \left[ \dfrac{c' \Delta l \cos \alpha + (W - F_v - u \Delta l \cos \alpha)\tan \phi'}{\cos \alpha + (\sin \alpha \tan \phi')/F} \right]}{\sum W \sin \alpha - M_n/r} \tag{6.78}$$

The term $M_n$ represents moments due to all known forces except the weight, including moments produced by the seismic forces ($kW$), external loads ($P$), and reinforcement ($T$) on the slice in Figure 6.12.

Equation (6.78) is the equation for the factor of safety by the Simplified Bishop procedure extended to include additional known forces like those due to seismic loads, reinforcement, and external water pressures. However, because only vertical and not horizontal force equilibrium is considered, the method largely neglects any contribution to the normal stresses on the slip surface from horizontal forces, such as a seismic force and horizontal reinforcement forces. Horizontal forces are included in Eq. (6.78) only indirectly through their contribution to the moment, $M_n$. Consequently, care should be exercised if the Simplified Bishop procedure is used where there are significant horizontal forces that contribute to stability. However, it has been the writers' experience that even when there are significant horizontal forces, the Simplified Bishop procedure produces results comparable to those obtained by procedures that satisfy all conditions of equilibrium.

The Simplified Bishop procedure is often used for analysis of reinforced slopes. If the reinforcement is horizontal, the reinforcement contributes in the equa-

[2]The weight, $W$, could also be included in the force, $F_v$, but for now the weight will be kept separate to make these equations more easily compared with those derived previously with no known forces except the slice weight.

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tion of moment equilibrium but does not contribute in the equation for equilibrium of forces in the vertical direction. Thus, the effect of the reinforcement can be neglected in the equation of vertical force equilibrium [Eq. (6.74)]. However, if the reinforcement is inclined, the reinforcement contributes to both moment equilibrium and vertical force equilibrium. Some engineers have ignored the contribution of inclined reinforcement in the equation of vertical force equilibrium [Eq. (6.74)], while others have included its effect. Consequently, different results have been obtained depending on whether or not the contribution of vertical reinforcement forces is included in the equation of vertical force equilibrium (Wright and Duncan, 1991). It is recommended that the contribution always be included, as suggested by Eq. (6.74), and when reviewing the work of others it should be determined whether or not the force has been included.

Equations similar to those presented above for the Simplified Bishop procedure can be derived using the Ordinary Method of Slices. However, because of its relative inaccuracy, the Ordinary Method of Slices is generally not used for analyses of more complex conditions, such as those involving seismic loading or reinforcement. Therefore, the appropriate equations for additional known loads with the Ordinary Method of Slices are not presented here.

### Complete Bishop Procedure

Bishop (1955) originally presented two different procedures for slope stability analysis. One procedure is the ''Simplified'' procedure described above; the other procedure considered all of the unknown forces acting on a slice and made sufficient assumptions to fully satisfy static equilibrium. The second procedure is often referred to as the *Complete Bishop procedure.* For the complete procedure, Bishop outlined what steps and assumptions would be necessary to fully satisfy static equilibrium; however, no specific assumptions or details were stated. In fact, Bishop's second procedure was similar to a procedure that Fellenius (1936) described much earlier. Neither of these ''procedures'' consists of a well-defined set of assumptions and steps like those of the other procedures described in this chapter. Because neither the Complete Bishop procedure nor Fellenius's rigorous method has been described completely, these methods are not considered further. Since the pioneering contributions of Bishop and Fellenius, several procedures have been developed that set forth a distinct set of assumptions and steps for satisfying static equilibrium. These newer procedures are discussed later in this chapter.

## PROCEDURES OF SLICES: NONCIRCULAR SLIP SURFACES

Up to this point, all of the procedures of slices as well as the single free-body procedures that have been presented are based on relatively simple shapes for the slip surface: a plane, a logarithmic spiral, or a circle. Many times the slip surface is more complex, often following zones or layers of relatively weak soil or weak interfaces between soil and other materials, such as geosynthetics. In such cases it is necessary to compute stability using more complex shapes for the slip surface. Several procedures have been developed for analyses of more complex, noncircular slip surfaces. These procedures are all procedures of slices. Some of the procedures consider all of the conditions of static equilibrium; others consider only some of them. Several procedures are based on satisfying only the requirements of force equilibrium; these are known as *force equilibrium procedures*. Most of the other procedures for analysis with noncircular slip surfaces consider all of the requirements for static equilibrium and are referred to as *complete equilibrium procedures*. The force equilibrium and complete equilibrium procedures are discussed separately below.

### Force Equilibrium (Only) Procedures

Force equilibrium procedures satisfy only the conditions of force equilibrium and ignore moment equilibrium. These procedures use the equations for equilibrium of forces in two mutually perpendicular directions to compute the factor of safety, the forces on the base of each slice, and the resultant interslice forces. Usually, forces are resolved either vertically and horizontally or parallel and perpendicular to the base of the slice. To obtain a solution that is statically determinate (equal number of equations and unknowns), the inclinations of the forces between each slice are assumed. Once the inclination of the force is assumed, the factor of safety can be calculated.

**Interslice force assumptions.** Various authors have suggested different assumptions for the interslice force inclinations to be used in force equilibrium procedures. Three of the most recognized assumptions are summarized in Table 6.1 along with the names commonly associated with each assumption.

Of the three assumptions shown in Table 6.1, the one suggested by Lowe and Karafiath (1959), that the interslice forces act at the average of the inclination of the slope and slip surface, seems to produce the best results. Factors of safety calculated using Lowe and Karafiath's procedure are generally in closest agree-

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**Table 6.1 Interslice Force Assumptions Used in Force Equilibrium Procedures**

| Procedure/assumption | Description |
|---|---|
| Lowe and Karafiath (Lowe and Karafiath, 1959) | The interslice forces are assumed to be inclined at the average slope of the ground surface and slip surface. The inclination varies from slice to slice, depending on where the slice boundaries are located. |
| Simplified Janbu (Janbu et al., 1956; Janbu, 1973) | The side forces are horizontal; there is no shear stress between slices. Correction factors are used to adjust (increase) the factor of safety to more reasonable values. |
| U.S. Army Corps of Engineers' modified Swedish method (U.S. Army Corps of Engineers, 1970). | The side forces are parallel to the average embankment slope. Although not clearly stated in their 1970 manual, the Corps of Engineers has established that the interslice force inclination will be the same for all slices[a]. |

[a]During the development of the UTEXAS2 and UTEXAS3 slope stability software, the U.S. Army Corps of Engineers' CAGE Committee made the decision that in the Modified Swedish procedure, all side forces would be assumed to be parrallel.

ment with the factors of safety calculated using procedures that satisfy complete equilibrium.

The Simplified Janbu procedure is based on the assumption that the interslice forces are horizontal. This assumption alone almost always produces factors of safety that are smaller than those obtained by more rigorous procedures that satisfy complete equilibrium. To account for this, Janbu et al. (1956) proposed the correction factors shown in Figure 6.13. These correction factors are based on a number of slope stability computations using both the simplified procedure with horizontal interslice forces and the more rigorous GPS procedure described later. The correction factors are only approximate, being based on analyses of 30 to 40 cases; however, the correction factors seem to provide an improved value for the factor of safety for many slopes.[3] Caution should be used in evaluating analyses that are reported using the Simplified Janbu procedure: Some analyses and computer programs automatically apply the correction factor to the computed factor of safety; others do not. Whenever results are reported for the Simplified Janbu procedure, it should be determined whether the correction factor has been applied, as the correction can have a noticeable effect on the results.

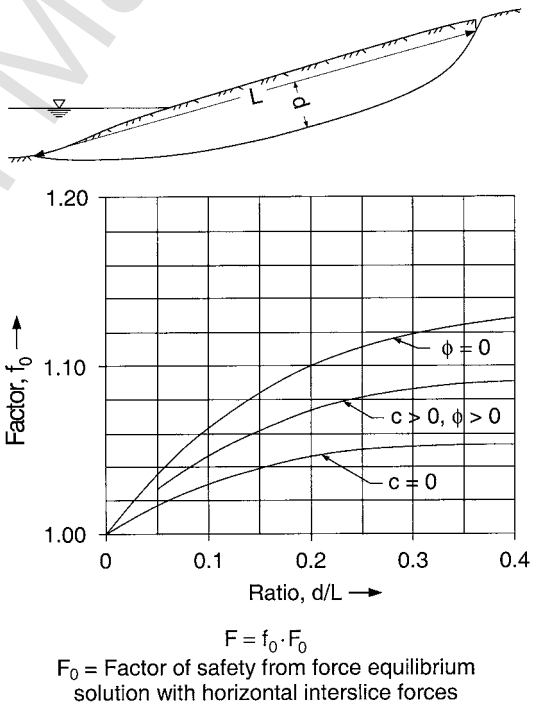



**Figure 6.13** Correction factors for Janbu's simplified procedure of slices.

The U.S. Army Corps of Engineers' Modified Swedish procedure is based on the assumption that the interslice forces act at the ''average inclination of the embankment slope'' (U.S. Army Corps of Engineers, 1970). This can be interpreted in at least three different ways, as illustrated in Figure 6.14. As shown in the

[3]Janbu (1973) indicates that the correction factors are based on a comprehensive investigation of some 40 different soil profiles. In a more recent personal communication (2003) between the authors and Janbu, Janbu confirmed that the correction factors were established based on analyses of approximately 30 cases in the files of the Norwegian Geotechnical Institute.

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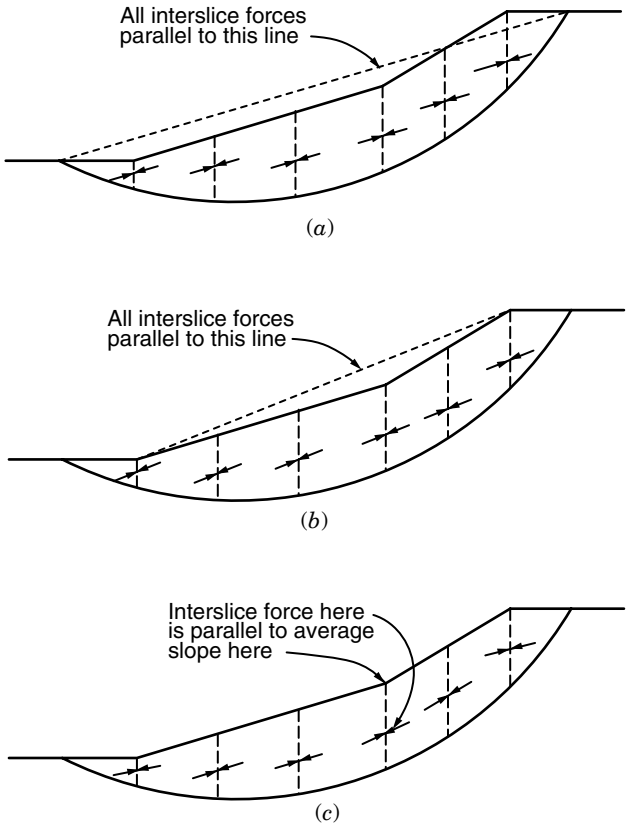



**Figure 6.14** Candidate interpretations of the U.S. Army Corps of Engineers' assumption for the interslice force inclinations in the Modified Swedish procedure—average inclinations of the embankment slope: (*a*) interpretation 1; (*b*) interpretation 2; (*c*) interpretation 3.

figure, the interslice force inclinations may be interpreted either as being the same for every slice (Figure 6.14*a* and *b*) or differing from slice to slice (Figure 6.14*c*). To the writers' knowledge all three interpretations that are shown in Figure 6.14 have been used. However, currently the standard practice is to assume that the interslice forces all have the same inclination.[4] Regardless of the interpretation, any of the three assumptions illustrated in Figure 6.14 can lead to factors of safety that are larger than those obtained by more rigorous procedures of analysis that fully satisfy moment equilibrium, and thus the factors of safety are unconservative. Accordingly, some engineers elect to assume flatter angles for the interslice forces than those suggested by the U.S. Army Corps of Engineers (1970).[5] If in the extreme the interslice forces are assumed to be horizontal, the Modified Swedish procedure becomes identical to the Simplified Janbu procedure (without the correction factor) and the procedure then probably underestimates the factor of safety. As with the Simplified Janbu procedure, it is recommended that the details of the interslice force assumptions be determined whenever reviewing the results of calculations performed by others using the Modified Swedish procedure.

One of the principal limitations of force equilibrium procedures is that the procedures are sensitive to what is assumed for the interslice force inclination. To illustrate this sensitivity, calculations were performed for the short-term stability of the homogeneous slope shown in Figure 6.15*a*. The slope is composed of saturated clay. The undrained shear strength is 400 psf at the elevation of the crest of the slope and increases linearly with depth below the crest at the rate of 7.5 psf per foot of depth. Stability calculations were performed using force equilibrium procedures with parallel interslice forces (i.e., all interslice forces had the same inclination). The interslice force inclination was varied from horizontal to 21.8°. An inclination of 21.8° represents interslice forces parallel to the slope face—the Corps of Engineers' Modified Swedish assumption. For each interslice force inclination assumed, the minimum factor of safety was computed using circular slip surfaces for simplicity. The factors of safety computed are plotted in Figure 6.15*b* versus the assumed interslice force inclination. The values range from approximately 1.38 to 1.74, a difference of approximately 25%. As discussed earlier, there is a unique value for the factor of safety calculated from moment equilibrium that satisfies complete static equilibrium for circular slip surfaces when $\phi = 0$. The factor of safety for moment equilibrium is 1.50. The factors of safety computed from the force equilibrium solutions where the interslice force inclinations were varied range from approximately 8% less (Simplified Janbu without correction factor) to 16% greater (Corps of Engineers Modified Swedish procedure) than the value satisfying complete static equilibrium. These differences are for the relatively simple slope and problem chosen for illustration; even larger differences should be anticipated for other slopes and soil properties.

The factor of safety computed by Spencer's procedure, which is described later in this chapter, is also

[4] During development of the UTEXAS2 and UTEXAS3 slope stability software, a decision was made by the Corps of Engineers' CAGE Committee that all interslice forces would be assumed to be parallel.

[5] In discussions with various Corps of Engineers' personnel, the writers have learned that flatter angles have been used at the engineers' discretion in recognition of the fact that steeper angles may lead to too high a value for the factor of safety.

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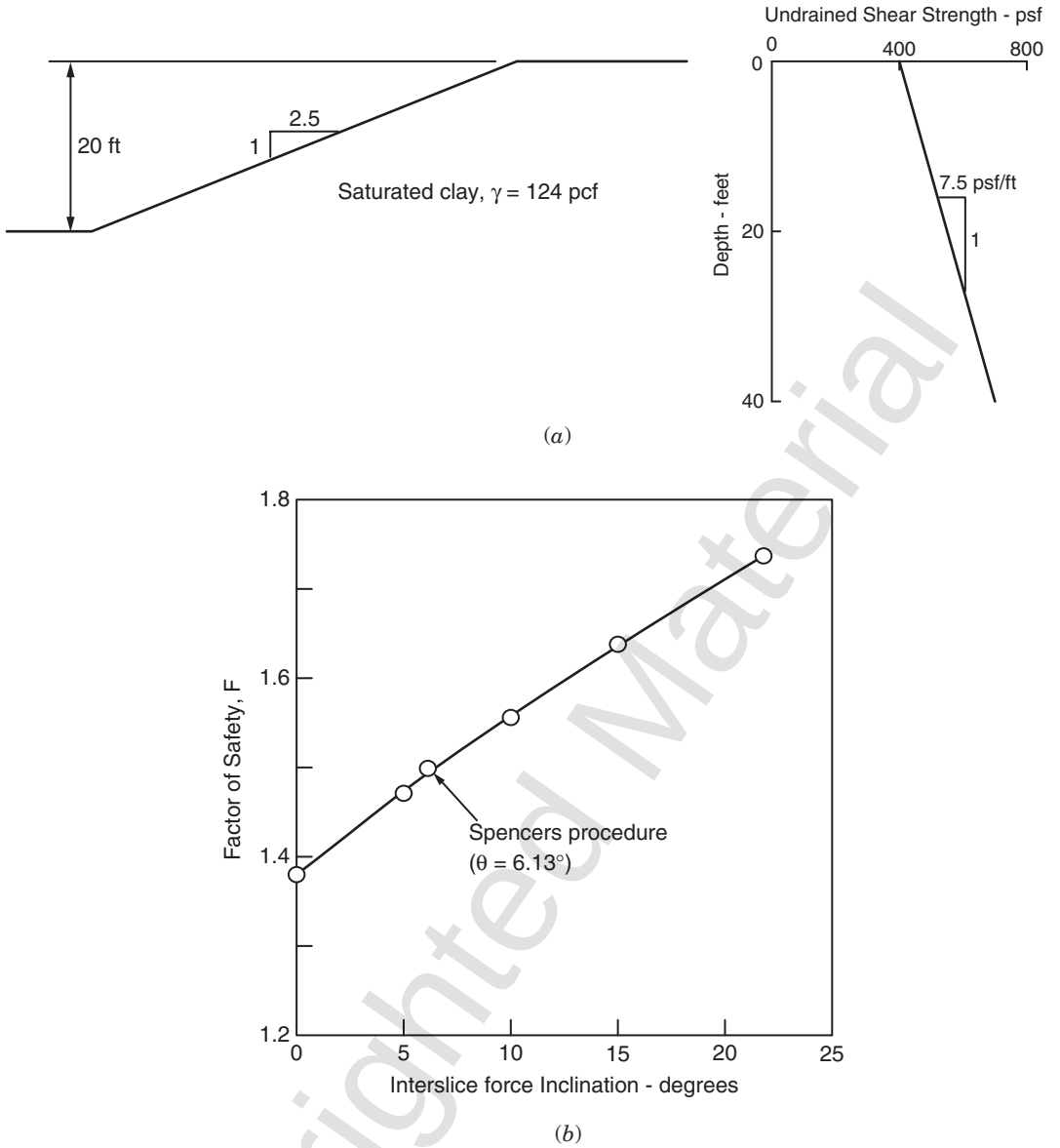



**Figure 6.15** Influence of interslice force inclination on the computed factor of safety for force equilibrium with parallel interslice forces.

plotted in Figure 6.15*b*. Spencer's procedure assumes that the interslice forces are parallel but solves for the interslice force inclination that satisfies *both* force and moment equilibrium. For this problem the minimum factor of safety by Spencer's procedure is 1.50, as expected, and the corresponding interslice force inclination is 6.11°. The factor of safety is plotted in Figure 6.15*b* versus the interslice force inclination (6.13°). Somewhat surprisingly, the factor of safety by Spencer's procedure, which satisfies complete equilibrium, is larger than what would be expected for a force equilibrium solution for a interslice force inclination of 6.13°; the point corresponding to Spencer's procedure plots slightly above the line from the force equilibrium solutions. The difference occurs because the critical circle for Spencer's procedure, where the interslice force inclination may be different for each circle, is not the same as the critical circle for the force equilibrium solution where the interslice force inclination is the same for all circles.

**Solution procedure: general.** All of the force equilibrium procedures require that trial-and-error methods be used to solve the equilibrium equations and calculate the factor of safety. Calculations using force equi-

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librium procedures have been performed for many years, and the procedures were first used long before electronic calculators and computers were readily available. Thus, originally the equilibrium equations were ''solved'' using trial-and-error graphical methods rather than numerical methods. The graphical methods consisted of constructing force equilibrium polygons (vector diagrams) repeatedly until the assumed factor of safety satisfied equilibrium (i.e., the force polygons ''closed''). Such graphical procedures are now seldom used and, instead, have been replaced by hand calculators, spreadsheets, or more sophisticated software programs. However, the graphical methods provide a useful insight into the force equilibrium procedures and the results that are obtained. In some instance, construction of the force equilibrium polygons is helpful in understanding the forces acting to stabilize or destabilize a slope and can even provide insight into numerical problems that may develop. Graphical methods are also helpful in explaining the force equilibrium procedures.

**Graphical solutions.** A graphical solution for the factor of safety is begun by assuming a trial value for the factor of safety. Once a trial value is assumed, the equilibrium requirements for the first slice are used to determine the magnitudes of the unknown normal force on the base of the slice and the interslice force between the first and second slices. It is convenient to illustrate this procedure with the example shown in Figures 6.16, 6.17, and 6.18. The equilibrium force polygon for the first slice is shown in Figure 6.16. The forces on the first slice include:

1. The weight for the slice ($W_1$).
2. A force ($u\ \Delta l$) representing the effect of the pore water pressure on the base of the slice, which is considered separately from the effective normal force.
3. A force ($R'$) representing the resultant due to the effective normal force ($N'$) and the component of the shear force due to friction ($N' \tan \phi'_d$). The resulting force acts at an angle of $\phi'_d$ from a line perpendicular to the base of the slice. Because the factor of safety has been assumed, $\phi'_d$ is defined.
4. A force due to the mobilized cohesion ($c'_d \Delta l$).
5. A interslice force ($Z_2$) on the right side of the slice acting at an inclination, $\theta$. The value of $\theta$ is assumed before starting a solution.

All of the information about the forces is known except for the magnitudes of the resulting forces $R'$ on the slip surface and $Z_2$ on the vertical slice boundary. The directions of both of these forces are known. From the equilibrium force polygon the magnitudes of $R'$ and $Z_2$ are then determined by the requirement that the force polygon must close. Once $R'$ is found, the effective normal force, $N'$, is simply calculated from $N' = R' \cos \phi'_d$.

For the second slice an undrained shear strength expressed in terms of total stresses is used. Because the soil is saturated, $\phi$ is zero. Thus, while effective stresses were used for the first slice, total stresses are used for the second slice. The forces on the second slice include:

1. The weight for the slice ($W_2$).
2. A force due to the mobilized cohesion ($c_d\ \Delta l$). Again, because a factor of safety has been assumed, $c_d$ can be calculated.
3. A force, $R$, representing the resulting force due to the normal force, $N$, and the component of shear strength due to friction. However, because $\phi = 0$, the resultant force, $R$, is the same as the normal force, $N$.
4. The interslice forces ($Z_2$ and $Z_3$) on the left and right sides of the slice, respectively.

Again the only two unknown quantities are the magnitudes of the force, $N$, on the base of the slice and the force, $Z_3$, on the right side of the slice. Thus, the magnitude of $N$ and $Z_3$ can be determined.

The steps above are repeated, constructing equilibrium force polygons slice-by-slice until the last slice is reached. For the last slice only the value of the resulting force, $R$, is unknown because there is no interslice force on the right of this slice. A value for this force ($R$) may or may not be found that closes the force polygon. As shown in Figure 6.18, the force polygon can be made to close only by introducing an additional force. The magnitude of the additional force required to close the force polygon will depend on the direction assumed. If all interslice forces are parallel, the additional force is generally assumed to have the same inclination as the interslice forces; if the inclination of the interslice forces varies from slice to slice, the imbalanced force is often assumed to be horizontal. In Figure 6.18 the additional force is assumed to be horizontal and is represented as $Z_{\text{imbalance}}$. The horizontal force indicates that the factor of safety that was assumed at the outset is not correct. The force $Z_{\text{imbalance}}$ provides a measure of the error in the assumed factor of safety. In the case of a slope facing to the left, like the one shown in Figure 6.18, if the imbalanced force acts to the left, it indicates that an additional force (one pushing the potential slide mass downslope) is needed

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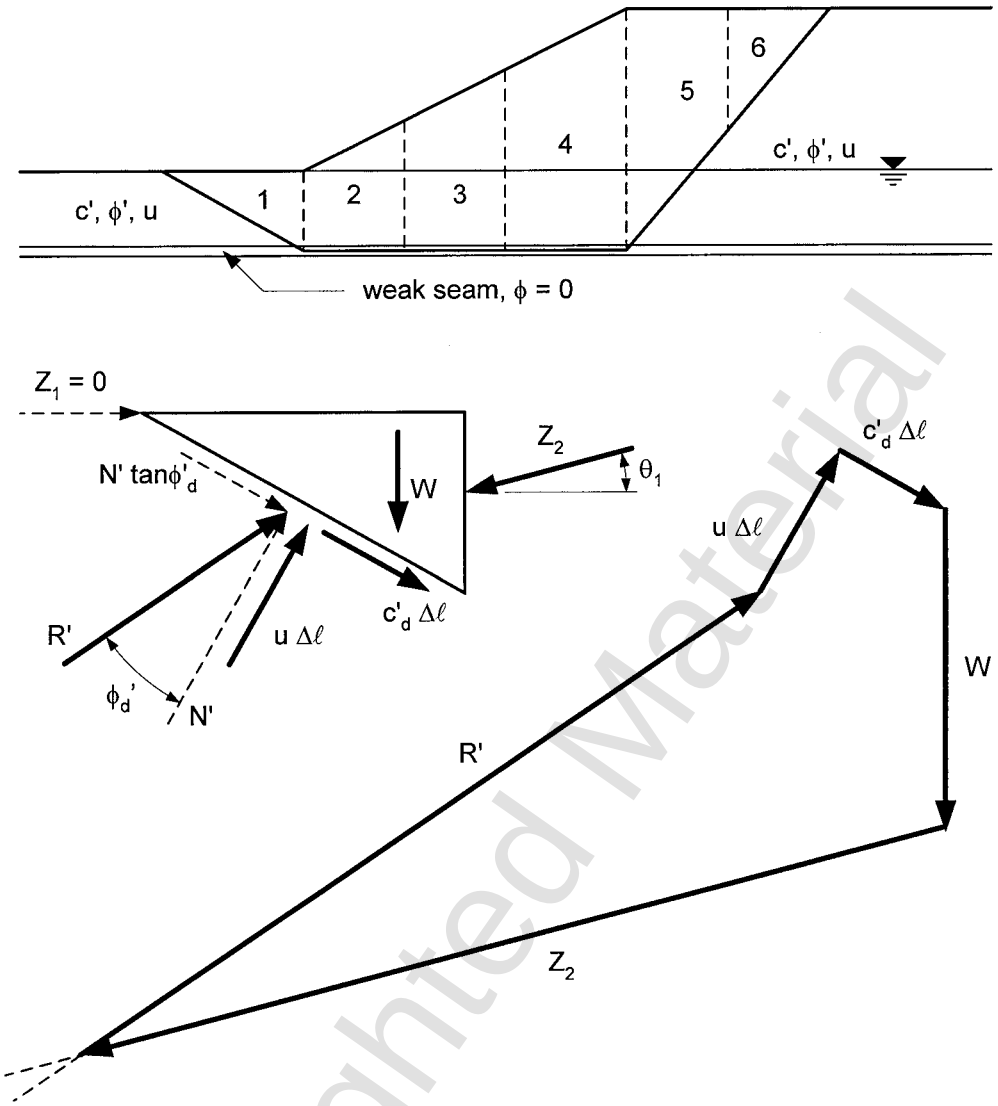



**Figure 6.16** Force equilibrium polygon (vector diagram) of forces acting on the first slice for a force equilibrium solution by the graphical method.

to produce the factor of safety that was assumed. Because such a force does not actually exist, the value that was assumed for the factor of safety must have been too low, and a larger value of $F$ should be assumed for the next trial.

In the graphical procedure, values for the factor of safety are assumed repeatedly and the equilibrium force polygons are drawn. This process is repeated until closure of the force polygons with negligible error (force imbalance) is achieved (i.e., until $Z_{\text{imbalance}} = 0$).

Although today, graphical procedures have largely been abandoned in favor of electronic means of calculation, the graphical procedures provide useful insight into the forces contributing to slope stability. The relative magnitudes of forces, like those due to cohesion and friction, for example, provide insight into the relative importance of each. Even though force polygons may no longer be used as the means of solving for the factor of safety, they provide a useful means for examining a solution graphically.

**Analytical solutions.** Today, most analyses performed using force equilibrium procedures are carried out by performing calculations using a spreadsheet or other computer program. In this case the equilibrium equations are written as algebraic equations. Consider the slice shown in Figure 6.19. Summation of forces in the vertical direction for an individual slice produces the following equilibrium equation:

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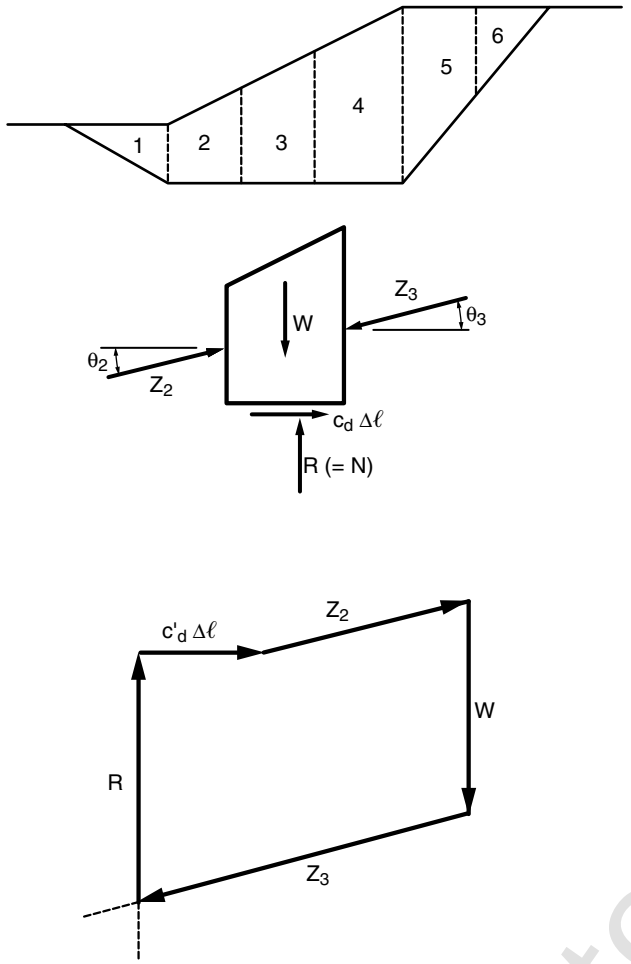



**Figure 6.17** Force equilibrium polygon (vector diagram) of forces acting on the second slice for a force equilibrium solution by the graphical method.

$$F_v + Z_i \sin \theta_i - Z_{i+1} \sin \theta_{i+1} + N \cos \alpha + S \sin \alpha = 0 \qquad (6.79)$$

where $Z_i$ and $\theta_i$ represent the respective magnitudes and inclinations of the inteslice force at the left of the slice, $Z_{i+1}$ and $\theta_{i+1}$ represent the corresponding values at the right of the slice, and $F_v$ represents the sum of all known forces in the vertical direction, including the weight of the slice. In the absence of any surface loads and reinforcement forces, $F_v$ is equal to $-W$. Forces are considered positive when they act upward. Summation of forces in the horizontal direction yields the following, second equation of force equilibrium:

$$F_h + Z_i \cos \theta_i - Z_{i+1} \cos \theta_{i+1} - N \sin \alpha + S \cos \alpha = 0 \qquad (6.80)$$

The quantity $F_h$ represents the net sum of all known forces acting on the slice in the horizontal direction; forces acting to the right are considered positive. If there are no seismic forces, external loads or reinforcement forces, the force, $F_h$, will be zero; for seismic loading alone, $F_h = -kW$.

Equations (6.79) and (6.80) can be combined with the Mohr–Coulomb equation for the shear force [Eq. (6.63)] to eliminate the shear and normal forces ($S$ and $N$) and obtain the following equation for the interslice force, $Z_{i+1}$, on the right side of a slice:

$$Z_{i+1} = \frac{\begin{array}{l} F_v \sin \alpha + F_h \cos \alpha + Z_i \cos(\alpha - \theta) \\ \quad -[F_v \cos \alpha - F_h \sin \alpha + u\, \Delta l \\ \quad + Z_i \sin (\alpha - \theta)](\tan \phi'/F) + c'\, \Delta l/F \end{array}}{\cos(\alpha - \theta_{i+1}) + [\sin(\alpha - \theta_{i+1}) \tan \phi']/F} \qquad (6.81)$$

By first assuming a trial value for the factor of safety, Eq. (6.81) is used to calculate the interslice force, $Z_{i+1}$, on the right of the first slice where $Z_i = 0$. Proceeding to the next slice, where $Z_i$ is equal to the value of $Z_{i+1}$ calculated for the previous slice, the interslice force on the right of the second slice is calculated. This process is repeated slice by slice for the rest of the slices from left to right until a force on the right of the last slice is calculated. If the force, $Z_{i+1}$, on the right of the last slice is essentially zero, the assumed factor of safety is correct because there is no ''right side'' on the last slice, which is triangular. If the force is not zero, a new trial value is assumed for the factor of safety and the process is repeated until the force on the right of the last slice is acceptably small.

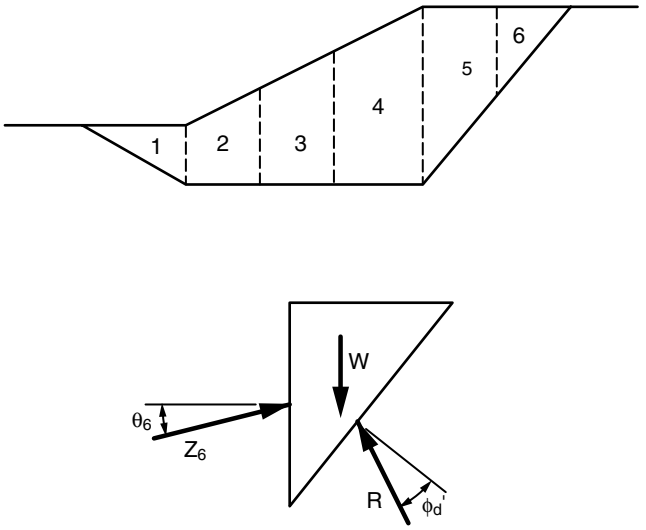
**Janbu's Generalized Procedure of Slices.** At this point it is appropriate to return to the procedure known as Janbu's Generalized Procedure of Slices (GPS) (Janbu, 1954, 1973*a*). There has been some debate as to whether this procedure satisfies complete equilibrium or only force equilibrium. In the GPS procedure the vertical components of the interslice forces are assumed based on a numerical approximation of the following differential equation for equilibrium of moments for a slice of infinitesimal width[6]:

$$X = -E \tan \theta_t + h_t \frac{dE}{dX} \qquad (6.82)$$

[6] Additional terms appear in this equation when there are external forces and other known forces acting on the slice; these additional terms are omitted here for simplicity.

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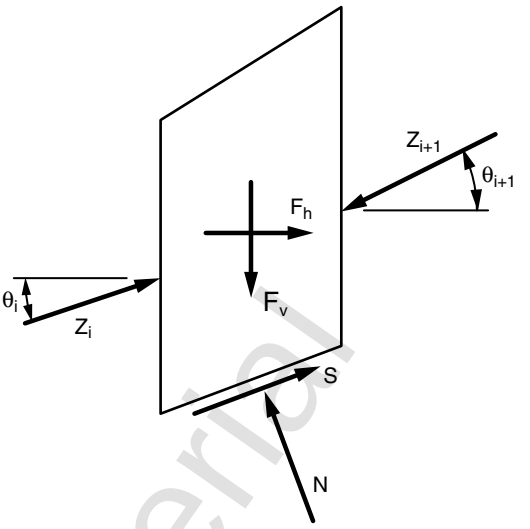




**Figure 6.19** Slice with forces for force equilibrium procedures.

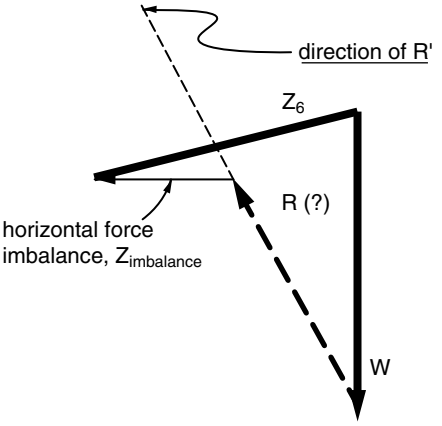



**Figure 6.18** Force equilibrium polygon (vector diagram) of forces acting on the last slice for a force equilibrium solution by the graphical method.

The quantities $X$ and $E$ represent the vertical and horizontal components, respectively, of the interslice forces. The quantity $h_t$ represents the height of the line of thrust above the slip surface. The line of thrust is the imaginary line drawn through the points where the interslice forces, $E$ (or $Z$), act (Figure 6.20). The term $\theta_t$ is an angle, measured from the horizontal, that represents the slope of the line of thrust. In the GPS procedure the location of the line of thrust is assumed by the user. The derivative $dE/dx$ in Eq. (6.82) is approximated numerically in the GPS procedure, and Eq. (6.82) is written in difference form as

$$X = -E \tan \theta_t + h_t \frac{E_{i+1} - E_{i-1}}{X_{i+1} - X_{i-1}} \qquad (6.83)$$

Equation (6.83) is based on considerations of moment equilibrium. However, Eq. (6.83) does not rigorously satisfy moment equilibrium in this discrete form; only Eq. (6.82) rigorously satisfies moment equilibrium. The other procedures of slices that are considered next satisfy complete equilibrium rigorously for a discrete set of slices. Thus, they are considered complete equilibrium procedures, whereas the GPS procedure is not.

The factor of safety is computed in the GPS procedure by performing successive force equilibrium solutions similar to those described in previous sections. Initially, the interslice forces are assumed to be horizontal and the unknown factor of safety and horizontal interslice forces, $E$, are calculated. Using this initial set of interslice forces, $E$, new interslice shear forces, $X$, are calculated from Eq. (6.83) and the force equilibrium solution is repeated. This process is repeated, each time making a revised estimate of the vertical component ($X$) of the interslice force and calculating the unknown factor of safety and horizontal interslice forces, until the solution converges (i.e., until there is not a significant change in the factor of safety). The GPS procedure frequently produces a factor of safety that is nearly identical to values calculated by procedures that rigorously satisfy complete static equilibrium. However, the procedure does not always produce a stable numerical solution that converges within an acceptably small error.

The GPS procedure satisfies moment equilibrium in only an approximate way [Eq. (6.83) rather than (6.82)]. It can be argued that once the approximate solution is obtained, a solution can be forced to satisfy moment equilibrium by summing moments for each

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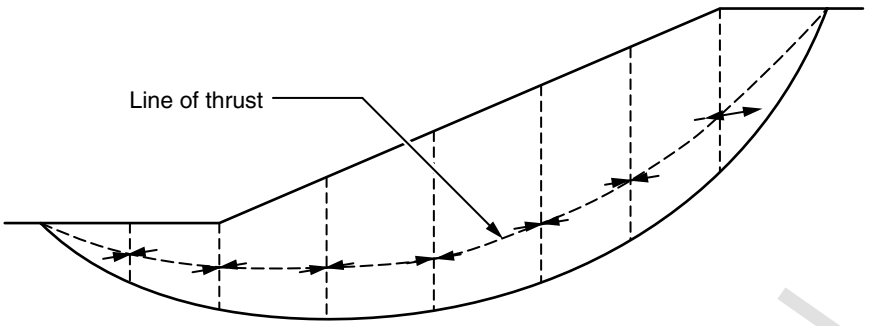



**Figure 6.20** Line of thrust describing the locations of the interslice forces on slice boundaries.

slice individually and calculating a location for the normal force ($N$) on the base of the slice, which will then satisfy moment equilibrium rigorously. This, however, can be done with any of the force equilibrium procedures described in this chapter, but by summing moments only after a factor of safety is calculated, there is no influence of moment equilibrium on the computed factor of safety. Summing moments to compute the location of the normal force on the base of slices does not appear to be particularly useful.

### Complete Equilibrium Procedures

Several different procedures of slices satisfy static equilibrium completely. Each of these procedures makes different assumptions to achieve a statically determinate solution. Several of these procedures are described in this section.

**Spencer's procedure.** Spencer's (1967) procedure is based on the assumption that the interslice forces are parallel (i.e., all interslice forces have the same inclination). The specific inclination of the interslice forces is unknown and is computed as one of the unknowns in the solution of the equilibrium equations. Spencer's procedure also assumes that the normal force ($N$) acts at the center of the base of each slice. This assumption has negligible influence on the computed values for the unknowns provided that a reasonably large number of slices is used; virtually all calculations with Spencer's procedure are performed by computer and a sufficiently large number of slices is easily attained.[7]

Spencer originally presented his procedure for circular slip surfaces, but the procedure is easily extended to noncircular slip surfaces. Noncircular slip surfaces are assumed here. In Spencer's procedure, two equilibrium equations are solved first. The equations represent overall force and moment equilibrium for the entire soil mass, consisting of all slices.[8] The two equilibrium equations are solved for the unknown factor of safety, $F$, and interslice force inclination, $\theta$. Because the interslice forces are assumed to be parallel, there is only one unknown inclination for interslice forces to be solved for.

The equation for force equilibrium can be written as

$$\sum Q_i = 0 \tag{6.84}$$

where $Q_i$ is the resultant of the interslice forces, $Z_i$ and $Z_{i+1}$, on the left and right, respectively, of the slice (Figure 6.21). That is,

$$Q_i = Z_i - Z_{i+1} \tag{6.85}$$

Because the interslice forces are assumed to be parallel, $Q_i$, $Z_i$, and $Z_{i+1}$ have the same direction and $Q_i$ is simply the scalar difference between the interslice forces on the left and right of the slice.

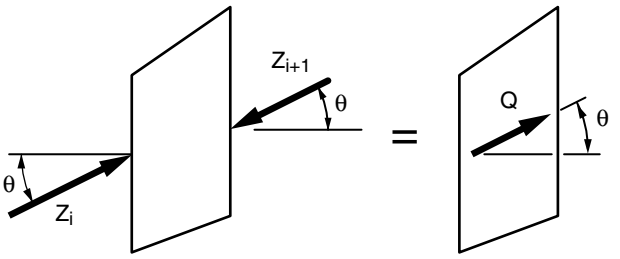



**Figure 6.21** Interslice forces and resultant when interslice forces are parallel.

[7] The number of slices used is discussed further in Chapter 14 and is not a matter of importance here.

[8] The equations for overall force equilibrium in the horizontal and vertical directions reduce to a single equation when interslice forces are parallel. Thus, there is only one force equilibrium equation considered at this stage.

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For moment equilibrium, moments can be summed about any arbitrary point. Taking moments about the origin ($x = 0$, $y = 0$) of a Cartesian coordinate system, the equation for moment equilibrium is expressed as

$$\sum Q(x_b \sin \theta - y_Q \cos \theta) = 0 \qquad (6.86)$$

where $x_b$ is the $x$ (horizontal) coordinate of the center of the base of the slice and $y_Q$ is the $y$ (vertical) coordinate of the point on the line of action of the force, $Q$, directly above the center of the base of the slice (Figure 6.22). The coordinate $y_Q$ can be expressed in terms of the $y$ coordinate of the point on the center of the base of the slice ($y_b$) by

$$y_Q = y_b + \frac{M_o}{Q \cos \theta} \qquad (6.87)$$

where $M_o$ is the moment produced by any known forces about the center of the base of the slice. In the absence of forces due to seismic loads, loads on the surface of the slope, and any internal forces due to reinforcement, the moment $M_o$ is zero and $y_Q = y_b$[9]. Each of the quantities in the summation shown for Eq. (6.86) represents the value for an individual slice. The subscript $i$ has been omitted for simplicity and will be omitted in subsequent discussion with the understanding that the quantities $Q$, $x_b$, $y_b$, and so on, represent values for individual slices.

The expression for $Q$ in the equilibrium equations [(6.84) and (6.86)] is obtained from the equations of force equilibrium for individual slices (Figure 6.23). Summing forces in directions perpendicular and parallel to the base of the slice gives the following two equilibrium equations:

$$N + F_v \cos \alpha - F_h \sin \alpha - Q \sin (\alpha - \theta) = 0 \qquad (6.88)$$

$$S + F_v \sin \alpha + F_h \cos \alpha + Q \cos (\alpha - \theta) = 0 \qquad (6.89)$$

The quantities $F_h$ and $F_v$ represent *all* known horizontal and vertical forces on the slice, including the weight of the slice, seismic loads, forces due to distributed and concentrated surface loads, and reinforcement forces. Combining these two force equilibrium equations [(6.88) and (6.89)] with the Mohr–Coulomb equation for the shear force, $S$ [Eq. (6.63)] and solving for $Q$ gives

$$Q = \frac{-F_v \sin \alpha - F_h \cos \alpha - (c' \, \Delta l/F) + (F_v \cos \alpha - F_h \sin \alpha + u \, \Delta l)(\tan \phi'/F)}{\cos(\alpha - \theta) + [\sin (\alpha - \theta) \tan \phi'/F]} \qquad (6.90)$$

Equations (6.87) for $y_Q$ and (6.90) for $Q$ can be substituted into the equilibrium equations [(6.84) and (6.86)] to give two equations with two unknowns: the factor of safety, $F$, and the interslice force inclination,

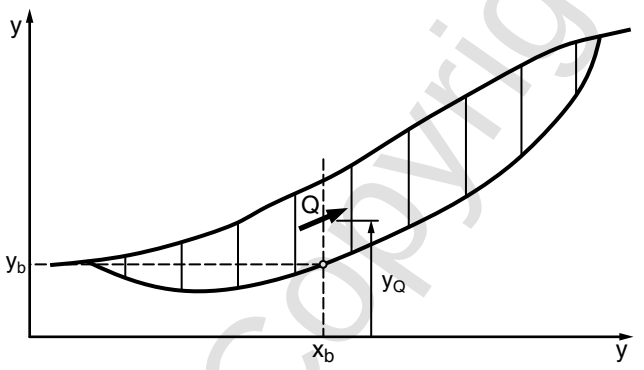



**Figure 6.22** Coordinates for noncircular slip surface used in Spencer's procedure.

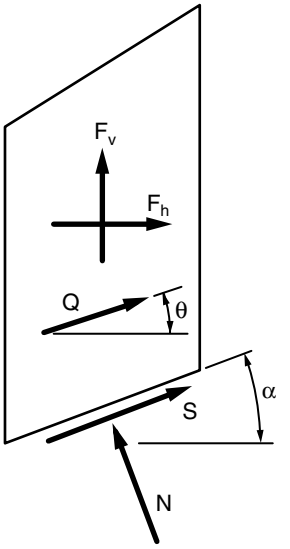



**Figure 6.23** Slice with all known and unknown forces for Spencer's procedure.

[9]The forces $W$, $S$, and $N$ all act through a common point on the center of the base of the slice, and thus $Q$ must also act through this point unless there are additional forces on the slice. In Spencer's (1967) original derivation $M_o$ is zero, and thus $y_Q = y_b$.

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$\theta$. Trial-and-error procedures are used to solve Eqs. (6.84) and (6.86) for $F$ and $\theta$. Values of $F$ and $\theta$ are assumed repeatedly until these two equations are satisfied within acceptable levels of error (force and moment imbalance). Once the factor of safety and interslice force inclination are computed, the equations of force and moment equilibrium for the individual slices are used to calculate the values of the normal force ($N$) on the base of the slice, the individual interslice force resultants ($Z$) between slices, and the location ($y_t$) of the interslice forces on the vertical boundary between the slices.

**Morgenstern and price procedure.** The Morgenstern and Price (1965) procedure assumes that the shear forces between slices are related to the normal forces as

$$X = \lambda f(x)E \tag{6.91}$$

where $X$ and $E$ are the vertical and horizontal forces between slices, $\lambda$ is an unknown scaling factor that is solved for as part of the unknowns, and $f(x)$ is an assumed function that has prescribed values at each slice boundary. In the Morgenstern and Price procedure the location of the normal force on the base of the slice is also explicitly or implicitly assumed. In the original formulation of the Morgenstern and Price procedure, stresses were integrated across each slice assuming that $f(x)$ varied linearly across the slice (Morgenstern and Price, 1967). This implicitly fixed the distribution of the normal stresses, including the location of the normal force on the base of the slice. In more recent implementations of the Morgenstern and Price procedure, discrete formulations have been used for slices and the location of the normal force has been assumed. Typically, the normal force is assumed to act at the midpoint of the base of the slice or at a point on the base of the slice that is directly below the center of gravity.

The unknowns that are solved for in the Morgenstern and Price procedure are the factor of safety ($F$), the scaling parameter ($\lambda$), the normal forces on the base of the slice ($N$), the horizontal interslice force ($E$), and the location of the interslice forces (line of thrust). The vertical component of the interslice force, $X$, is known [defined by Eq. (6.91)]; that is, once the unknowns are calculated using the equilibrium equations, the vertical component of the interslice forces is calculated from the independent equation (6.91).

Morgenstern and Price's procedure is similar to Spencer's procedure. The only difference in terms of unknowns is that Spencer's procedure involves a single interslice force inclination whereas Morgenstern and Price's procedure involves a single "scaling" parameter, $\lambda$. If the function $f(x)$ is assumed to be constant in Morgenstern and Price's procedure it produces results essentially identical to those using Spencer's procedure.[10] The major difference between the two procedures is that Morgenstern and Price's procedure provides added flexibility in the assumptions for the interslice force inclinations. The added flexibility allows the assumption regarding the interslice forces to be changed. However, the assumptions generally appear to have little effect on the computed factor of safety when static equilibrium is satisfied, and thus there is little practical difference among Spencer's, Morgenstern and Price's, and all the other complete equilibrium procedures of slices.

**Chen and Morgenstern procedure.** The Chen and Morgenstern (1983) procedure represents a refinement of the Morgenstern and Price procedure that attempts to account better for the stresses at the ends of a slip surface. Chen and Morgenstern suggested that at the ends of the slip surface the interslice forces must become parallel to the slope. This leads to the following relationship between the shear ($X$) and horizontal ($E$) forces on the side of the slice:

$$X = [\lambda f(x) + f_0(x)]E \tag{6.92}$$

where $f(x)$ and $f_0(x)$ are two separate functions that define the distribution of the interslice force inclinations. The function $f(x)$ is zero at each end of the slip surface, and the function $f_0(x)$ is equal to the tangent of the slope inclination at each end of the slip surface. The variations of both $f(x)$ and $f_0(x)$ between the two ends of the slip surface are assumed by the engineer. Chen and Morgenstern's procedure restricts the range of admissible interslice force inclinations and thus reduces the range of possible solutions.

**Sarma's procedure.** Sarma's (1973) procedure is different from all the other procedures discussed in this chapter because it considers the seismic coefficient ($k$) to be unknown, and the factor of safety is considered to be known. A value for the factor of safety is assumed and the seismic coefficient required to produce this factor of safety is solved for as an unknown. Usually, the factor of safety is assumed to be 1 and the seismic coefficient that is then calculated represents the seismic coefficient required to cause sliding, referred to in Chapter 10 as the *seismic yield coefficient.* In

[10] There may be subtle differences in the two procedures in this case because of slightly different locations assumed for the normal forces on the base of the slice. However, these differences are negligible for all practical purposes.

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Sarma's procedure the shear force between slices is related to shear strength by the relationship

$$X = \lambda f(x) S_v \tag{6.93}$$

where $S_v$ is the available shear force on the slice boundary, $\lambda$ is an unknown scaling parameter, and $f(x)$ is an assumed function with prescribed values at each vertical slice boundary. The shear force, $S_v$, depends on the shear strength parameters ($c$, $c'$ and $\phi$, $\phi'$) for the soil along the slice boundary and for frictional materials ($\phi$, $\phi' > 0$) on the normal (horizontal) interslice force, $E$. For effective stress analyses the shear force also depends on the pore water pressure on the slice boundary.

Sarma's procedure was developed for evaluations of seismic stability and offers some advantage over other procedures for this purpose. In Sarma's procedure the seismic coefficient and other unknowns can be calculated directly; no iterative, trial-and-error procedure is required to calculate the unknowns. Sarma's procedure can also be used to calculate a factor of safety by repeatedly assuming different values for the factor of safety and calculating the seismic coefficient. The process is repeated until the value assumed for the seismic coefficient matches the value for which the factor of safety is desired. For slopes with no seismic loads the target seismic coefficient is zero. However, to compute a factor of safety, Sarma's procedure requires trial and error and thus offers no advantage over other complete equilibrium procedures.

The function, $f(x)$, and scaling parameter, $\lambda$, in Sarma's procedure are similar, but not identical, to the corresponding quantities in the Morgenstern and Price (1965) and the Chen and Morgenstern (1983) procedures. Depending on the assumption for $f(x)$ in these procedures, different inclinations will be found for the interslice forces. Depending on the range of assumed $f(x)$ patterns, there will probably be some overlap among solutions by the three (Morgenstern and Price, Chen and Morgenstern, and Sarma) procedures. Except for small differences in the values for $f(x)$ and $\lambda$, the three procedures should produce similar results for either the seismic coefficient required to produce a given factor of safety or the factor of safety corresponding to a given seismic coefficient. Sarma's procedure is easier to use to calculate a seismic coefficient for a prescribed factor of safety. On the other hand, the Morgenstern and Price procedure involves an assumption for the interslice forces that is much simpler and easier to use.

Sarma's procedure requires that the shear strength, $S_v$, along vertical slice boundaries be determined. For complex slopes with several materials and complex distributions of pore water pressure, Sarma's procedure becomes relatively complex. For frictional materials ($\phi$, $\phi' > 0$), additional assumptions must then be made about what fractions of the total normal force ($E$) between slices is distributed to each different material along the slice boundary. If the shear strength is represented by effective stresses, the distribution of pore water pressures along the slice boundary must also be considered. This makes the procedure excessively complex for many practical problems and difficult to implement in computer software. The major utility of Sarma's procedure seems to be for hand calculations for slopes with relatively simple geometries.

**Discussion.** All of the complete equilibrium procedures of slices have been shown to give very similar values for the factor of safety (Fredlund and Krahn, 1977; Duncan and Wright, 1980). Thus, no complete equilibrium procedure is significantly more or less accurate than another. Spencer's procedure is the simplest of the complete equilibrium procedures for calculating the factor of safety, while Sarma's procedure may be simplest for calculating the seismic coefficient required to produce failure (i.e., the seismic yield coefficient).

Morgenstern and Price's and Chen and Morgenstern's procedures are the most rigorous and flexible of the complete equilibrium procedures and may be useful for cases where interslice forces might have a significant effect on stability. In most cases interslice force inclinations have little effect on the factor of safety computed, provided that complete equilibrium is satisfied. The writers are aware of few cases where the assumptions regarding interslice forces have a noticeable effect on either the computed factor of safety or the numerical stability of a solution in the complete equilibrium procedures. Two cases where the assumptions regarding interslice force inclinations can be important are:

1. When the slip surface is forced to change direction abruptly, due to the geometry and properties of the slope cross section (Figure 6.24*a*)
2. For slopes with significant forces due to reinforcement or external loads whose direction is very different from the usual direction of the interslice forces (Figure 6.24*b* and *c*).

In these two cases, procedures that will allow the interslice force assumptions to be varied are useful in

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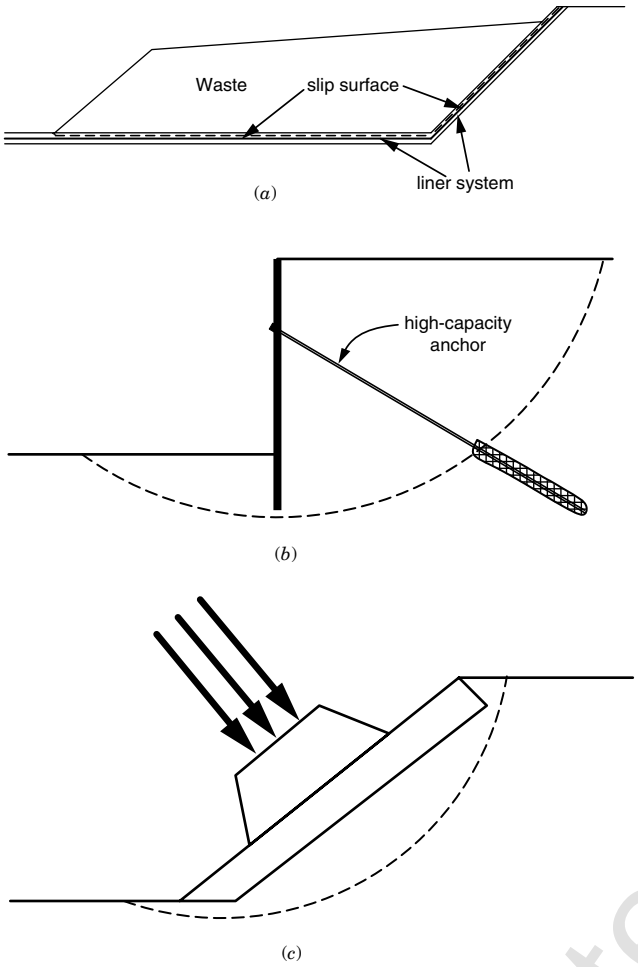



**Figure 6.24** Slope and slip surface conditions where the assumptions pertaining to the interslice forces may have a significant effect on the results of slope stability computations by complete equilibrium procedures.

establishing the amount of uncertainty and probable ranges in the factor of safety.

## ASSUMPTIONS, EQUILIBRIUM EQUATIONS, AND UNKNOWNS

As noted at the beginning of this chapter, all of the limit equilibrium procedures employ the equations of static equilibrium to compute a factor of safety. Assumptions are required to make the problem statically determinate and to obtain a balance between the number of equations and the number of unknowns that are solved for. Table 6.2 lists the various procedures discussed in this chapter along with the assumptions that are made, the equilibrium equations that are satisfied, and the unknowns. In each case there is an equal number of equations and unknowns.

Thirteen different procedures of limit equilibrium analysis have been discussed in this chapter. In general, the procedures that satisfy complete static equilibrium are the most accurate and preferred when all other things are equal. However, there are numerous instances where simpler, though less accurate procedures are useful. All of the procedures examined in this chapter are summarized in Table 6.3 along with the range or conditions of practical usefulness for each procedure.

## REPRESENTATION OF INTERSLICE FORCES (SIDE FORCES)

Up to this point the interslice forces have been assumed to represent all of the forces transmitted across a slice boundary, including the forces due to effective stresses in the soil, pressures in the pore water, and forces in any internal soil reinforcing. The interslice forces have been represented either in terms of their vertical and horizontal components ($X$ and $E$), or the resultant force and its inclination ($Z$ and $\theta$). No distinction has been made between the various components that make up the total force on an interslice boundary. However, because some of the forces, such as the force due to water pressures, may be known, it is possible to separate the forces into various components, which are then treated independently.

### Soil and Water Forces

For ordinary unreinforced slopes the interslice forces represent the forces due to effective stresses and pore water pressures. These forces can be represented separately as an effective force in the soil and a force in the water. Possible representations of the interslice forces are illustrated in Figure 6.25. When the forces are represented as total forces, as they have been in previous sections of the chapter, the shear and normal components, $X$ and $E$, respectively, are treated as unknowns and their values are either assumed or calculated from the equilibrium equations (Figure 6.25*b*). If, instead, the forces are represented by the forces due to effective stresses ($E'$ and $X$) plus the force due to water pressure ($U$), the water pressures are assumed to be known and the effective force components are treated as either unknowns that are calculated or they are assumed (Figure 6.25*c*). In either representation there are two forces that must be calculated or assumed on each slice boundary: $E$ and $X$, or $E'$ and $X$.

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**Table 6.2 Assumptions, Equilibrium Conditions, and Unknowns in Limit Equilibrium Procedures**

| Procedure | Assumptions | Equilibrium equations satisfied | Unknowns solved for |
|---|---|---|---|
| Infinite Slope | A slope of infinite extent; slip surface parallel to slope face. | 1 Σ Forces perpendicular to slope<br>1 Σ Forces parallel to slope<br>2 Total equations<br>(Moment equilibrium is implicitly satisfied) | 1 Factor of safety ($F$)<br>1 Normal force on shear surface ($N$)<br>2 Total unknowns |
| Logarithmic Spiral | The slip surface is a logarithmic spiral. | 1 Σ Moments about center of spiral<br>1 Total equations<br>(Force equilibrium is implicitly satisfied) | 1 Factor of safety ($F$)<br>1 Total unknown |
| Swedish Circle ($\phi = 0$) | The slip surface is circular; the friction angle is zero. | 1 Σ Moments about center of circle<br>1 Total equations<br>(Force equilibrium is implicitly satisfied) | 1 Factor of safety ($F$)<br>1 Total unknown |
| Ordinary Method of Slices (also known as Fellenius's Method; Swedish Method of Slices) | The slip surface is circular; the forces on the sides of the slices are neglected. | 1 Σ Moments about center of circle<br>1 Total equations | 1 Factor of safety ($F$)<br>1 Total unknown |
| Simplified Bishop | The slip surface is circular; the forces on the sides of the slices are horizontal (i.e., there is no shear force between slices). | 1 Σ Moments about center of circle<br>$n$ Σ Forces in the vertical direction.<br>$n + 1$ Total equations | 1 Factor of safety ($F$)<br>$n$ Normal force on the base of slices ($N$)<br>$n + 1$ total unknowns |
| Force Equilibrium (Lowe and Karafiath, Simplified Janbu, Corps of Engineer's Modified Swedish, Janbu's GPS procedure) | The inclinations of the interslice forces are assumed; assumptions vary with procedure. | $n$ Σ Forces in the horizontal direction<br>$n$ Σ Forces in the vertical direction<br>$2n$ Total equations | 1 Factor of safety ($F$)<br>$n$ Normal force on the base of slices ($N$)<br>$n - 1$ Resultant interslice forces ($Z$)<br>$2n$ Total unknowns |
| Spencer | Interslice forces are parallel, (i.e., all have the same inclination). The normal force ($N$) acts at the center of the base of the slice (typically). | $n$ Σ Moments about any selected point<br>$n$ Σ Forces in the horizontal direction<br>$n$ Σ Forces the vertical direction<br>$3n$ Total equations | 1 Factor of safety ($F$)<br>1 Interslice force inclination ($\theta$)<br>$n$ Normal force on the base of slices ($N$)<br>$n - 1$ Resultant interslice forces ($Z$)<br>$n - 1$ Location of side forces (line of thrust)<br>$3n$ Total unknowns |

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**Table 6.2 (*Continued*)**

| Procedure | Assumptions | Equilibrium equations satisfied | Unknowns solved for |
|---|---|---|---|
| Morgenstern and Price | Interslice shear force is related to interslice normal force by $X = \lambda f(x)E$; the normal force ($N$) acts at the center of the base of the slice (typically). | $n$ Σ Moments about any selected point<br>$n$ Σ Forces in the horizontal direction<br>$n$ Σ Forces in the vertical direction<br>$3n$ Total equations | 1 Factor of safety ($F$)<br>1 Interslice force inclination ''scaling'' factor ($\lambda$)<br>$n$ Normal force on the base of slices ($N$)<br>$n - 1$ Horizontal interslice forces ($E$)<br>$n - 1$ Location of interslice forces (line of thrust)<br>$3n$ Total unknowns |
| Chen and Morgenstern | Interslice shear force is related to interslice normal force by $X = [\lambda f(x) + f_o(x)]E$; the normal force ($N$) acts at the center of the base of the slice (typically). | $n$ Σ Moments about any selected point<br>$n$ Σ Forces in the horizontal direction<br>$n$ Σ Forces in the vertical direction<br>$3n$ Total equations | 1 Factor of safety ($F$)<br>1 Interslice force inclination ''scaling'' factor ($\lambda$)<br>$n$ Normal force on the base of slices ($N$)<br>$n - 1$ Horizontal interslice forces ($E$)<br>$n - 1$ Location of interslice forces (line of thrust)<br>$3n$ Total unknowns |
| Sarma | Interslice shear force is related to the interslice shear strength, $S_v$, by $X = \lambda f(x)S_v$; interslice shear strength depends on shear strength parameters, pore water pressures, and the horizontal component of interslice force; the normal force ($N$) acts at the center of the base of the slice (typically). | $n$ Σ Moments about any selected point<br>$n$ Σ Forces in the horizontal direction<br>$n$ Σ Forces in the vertical direction<br>$3n$ Total equations | 1 Seismic coefficient ($k$) [or factor of safety ($F$) if trial and error is used]<br>1 Interslice force scaling factor ($\lambda$)<br>$n$ Normal force on the base of slices ($N$)<br>$n - 1$ Horizontal interslice forces ($E$)<br>$n - 1$ Location of side forces (line of thrust)<br>$3n$ Total unknowns |

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**Table 6.3 Summary of Procedures for Limit Equilibrium Slope Stability Analysis and Their Usefulness**

| Procedure | Use |
|---|---|
| Infinite Slope | Homogeneous cohesionless slopes and slopes where the stratigraphy restricts the slip surface to shallow depths and parallel to the slope face. Very accurate where applicable. |
| Logarithmic Spiral | Applicable to homogeneous slopes; accurate. Potentially useful for developing slope stability charts and used some in software for design of reinforced slopes. |
| Swedish Circle; $\phi = 0$ method | Applicable to slopes where $\phi = 0$ (i.e., undrained analyses of slopes in saturated clays). Relatively thick zones of weaker materials where the slip surface can be approximated by a circle. |
| Ordinary Method of Slices | Applicable to nonhomogeneous slopes and $c$–$\phi$ soils where slip surface can be approximated by a circle. Very convenient for hand calculations. Inaccurate for effective stress analyses with high pore water pressures. |
| Simplified Bishop procedure | Applicable to nonhomogeneous slopes and $c$–$\phi$ soils where slip surface can be approximated by a circle. More accurate than Ordinary Method of Slices, especially for analyses with high pore water pressures. Calculations feasible by hand or spreadsheet. |
| Force Equilibrium procedures (Lowe and Karafiath's side force assumption recommended) | Applicable to virtually all slope geometries and soil profiles. The only procedures suitable for hand calculations with noncircular slip surfaces. Less accurate than complete equilibrium procedures and results are sensitive to assumed inclinations for interslice forces. |
| Spencer's procedure | An accurate procedure applicable to virtually all slope geometries and soil profiles. The simplest complete equilibrium procedure for computing the factor of safety. |
| Morgenstern and Price's procedure | An accurate procedure applicable to virtually all slope geometries and soil profiles. Rigorous, well-established complete equilibrium procedure. |
| Chen and Morgenstern's procedure | Essentially an updated Morgenstern and Price procedure. A rigorous and accurate procedure applicable to any shape of slip surface and slope geometry, loads, etc. |
| Sarma's procedure | An accurate procedure applicable to virtually all slope geometries and soil profiles. A convenient complete equilibrium procedure for computing the seismic coefficient required to produce a given factor of safety. Side force assumptions are difficult to implement for any but simple slopes. |

Fundamentally, it seems logical to represent the interslice forces by the known component due to water pressures and the unknown components due to effective stresses (Figure 6.25*c*). If the water pressures along an interslice boundary are simply hydrostatic, as suggested in Figure 6.26*a*, it is relatively easy to calculate the force due to water pressure and include it in stability computations. However, if the water pressures vary in a more complex manner, perhaps with distinctly different pressure regimes in different strata, as suggested in Figure 6.26*b*, it can be difficult to compute the force due to water pressures and its location. For complex groundwater conditions and subsurface soil profiles it is impractical to compute the forces due to water pressures, on each interslice boundary. Even though it may be possible to compute the force due to water pressures, the added effort complicates the logic and coding of computer programs that are used to perform such computations. For these practical reasons most slope stability formulations and computations are based on representing the interslice forces as total forces that include the force due to both the effective stress and the water pressures in the soil.

For procedures of slices that satisfy complete static equilibrium, it makes very little difference whether the unknown interslice forces include the water pressure force or the water pressure force is considered as a known force separately from the unknown force due to effective stress. The submerged slope shown in Figure 6.27 can be used to illustrate this. The factor of safety for this slope was calculated two different ways using Spencer's procedure of slices. In the first case the factor of safety was calculated with the interslice forces representing the total forces between slices. In the second case the water pressures on the side of the slice were computed and treated independent of the

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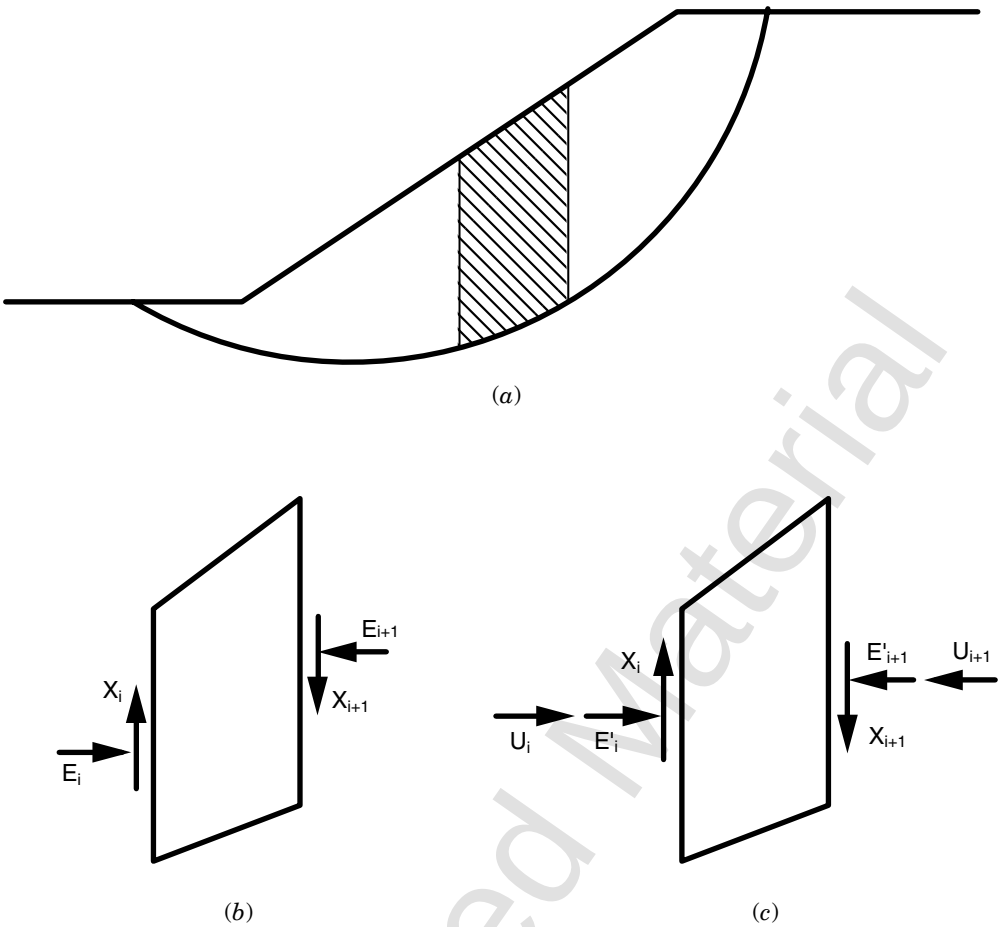



**Figure 6.25** (*a*) Slope and slice. Representations of interslice forces as (*b*) total forces and (*c*) effective forces with water pressure forces.

unknown interslice forces due to the effective stresses. Computations were performed for two different heights of water above the top of the slope: 30 and 60 ft. Computations were performed using both Spencer's complete equilibrium procedure and the Corps of Engineers' Modified Swedish force equilibrium procedure. For the Modified Swedish procedure the interslice forces (total or effective) were assumed to be parallel to the slope face (i.e., they were inclined at an angle of 21.8° from the horizontal). The factors of safety calculated for each case by both procedures are summarized in Table 6.4. The inclinations, $\theta$, of the interslice forces are also shown in Table 6.4. For Spencer's procedure it can be seen that the factors of safety calculated with total and effective interslice forces are almost identical. It can also be seen that the calculated inclination of the interslice forces is less when the interslice forces represent the total force (soil + water) rather than the effective force (soil only). The force due to water always acts horizontally, and it is therefore logical that the total force due to soil and water should be inclined at a flatter angle than the force due to effective stresses alone.

The factors of safety shown in Table 6.4 for the Modified Swedish procedure are very different depending on whether the interslice forces were represented using total or effective forces. Representation of the forces as effective forces with the water pressures treated separately produced factors of safety in close agreement with those calculated by Spencer's procedure, while representation of the interslice forces as total forces produced factors of safety that were from 47 to 60% higher! Representation of the interslice forces as total forces with the same inclination as the effective forces meant that the total forces were inclined more steeply than when the effective and water pressure forces were considered separately. As already shown, the steeper the inclination of the interslice forces, generally the higher the factor of safety (e.g., Figure 6.15). The results presented in Table 6.4 suggest

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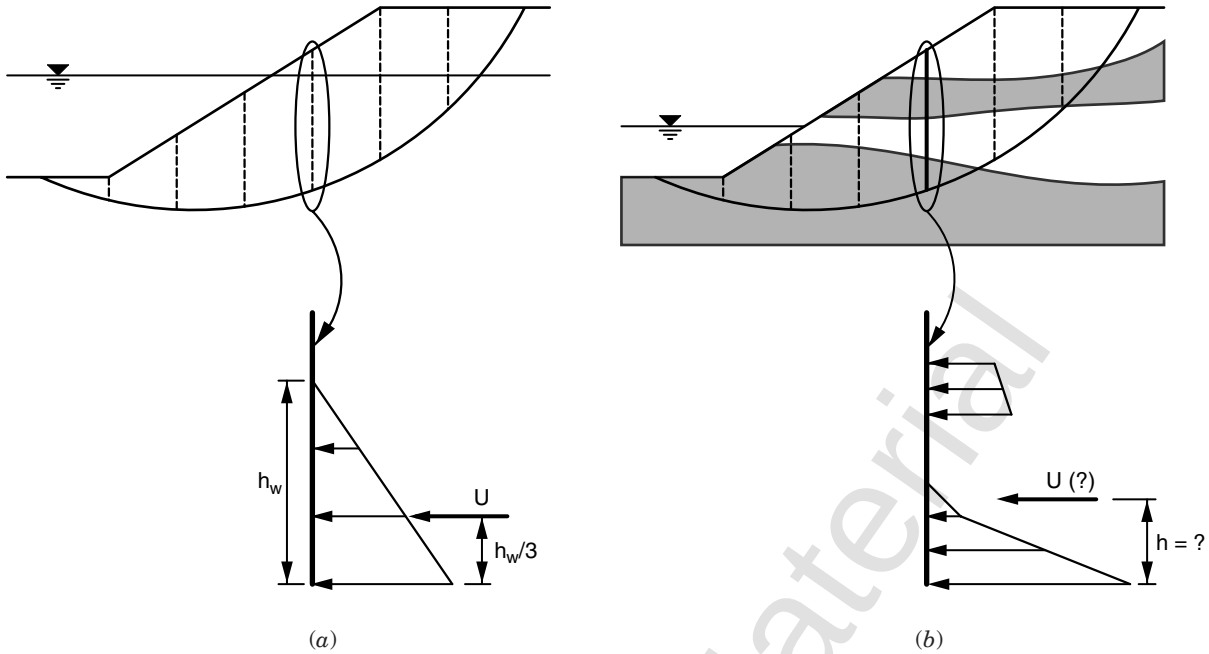



**Figure 6.26** Pore water pressure distributions on interslice boundaries; (*a*) simple hydrostatic pressures; (*b*) complex groundwater and pore water pressure conditions.

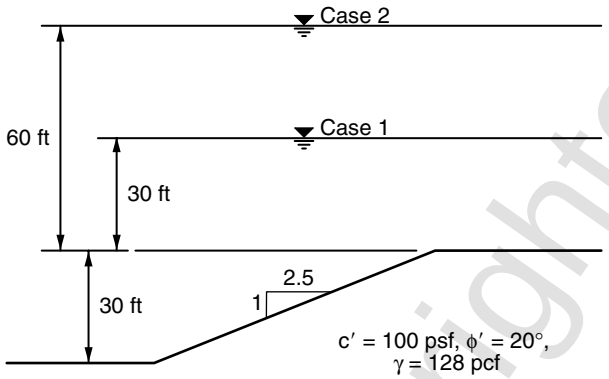



**Figure 6.27** Submerged slope analyzed with total and effective stress representations of the interslice forces.

that it would be better to express the interslice forces in terms of effective forces and forces due to water pressures. However, as discussed earlier, this is difficult and impractical for complex slopes.

### Soil-Water and Reinforcement Forces

For slopes like the one shown in Figure 6.28*a*, with reinforcement such as geogrids, geotextiles, piles, soil nails, and tieback anchors, the interslice forces include both the forces in the soil and water, as well as the forces transmitted across the interslice boundaries through the reinforcing elements. This allows additional choices pertaining to the representation of the interslice forces. One possibility is to let the interslice forces represent the forces both in the soil and in the reinforcing (Figure 6.28*b*); another possibility is to separate the interslice forces due to the reinforcement from the forces due to the soil and water (Figure 6.28*c*). If the interslice forces represent all the forces between the slices, the total interslice forces ($X$ and $E$, or $Z$ and $\theta$) are considered unknown. Once the interslice forces are calculated, the amount of force carried by just the soil and the water can be determined by subtracting the known forces due to the reinforcement from the total interslice forces. If, on the other hand, the reinforcement forces are considered separately from the forces due to the soil and the water, the reinforcement forces are determined first and treated as known interslice forces and the remaining (soil + water) interslice forces are treated as unknown forces. For hand calculations it is usually easiest (requires fewer calculations) to let the interslice forces represent all the force, soil + water + reinforcement, together as one set of forces. However, for calculations using a computer program, it is probably more appropriate to treat the forces in the reinforcement separately from the forces due to the soil and water.

To compute the stability of a reinforced slope in a computer program, it is necessary to define the forces

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**Table 6.4 Summary of Calculations for Slope Using Total Interslice Forces and Effective Interslice Forces with Water Pressures**[a]

| Procedure of analysis | 30 ft of water above slope crest | | 60 ft of water above slope crest | |
|---|---|---|---|---|
| | Total forces | Effective forces and water forces | Total forces | Effective forces and water forces |
| Spencer's | $F = 1.60$<br>$\theta = 1.7°$ | $F = 1.60$<br>$\theta = 17.2°$ | $F = 1.60$<br>$\theta = 1.1°$ | $F = 1.60$<br>$\theta = 17.2°$ |
| Corps of Engineers' Modified Swedish | $F = 2.38$<br>$\theta = 21.8°$ | $F = 1.62$<br>$\theta = 21.8°$ | $F = 2.60$<br>$\theta = 21.8°$ | $F = 1.62$<br>$\theta = 21.8°$ |

[a] $\theta$, inclination of interslice forces.

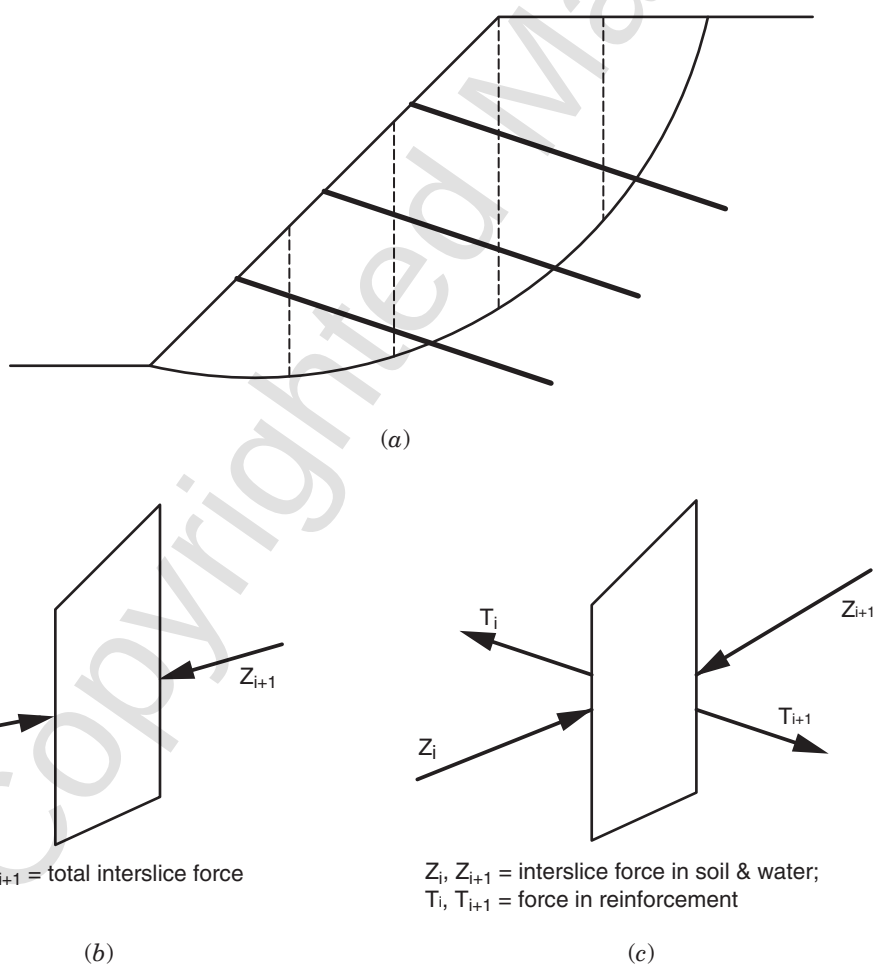



**Figure 6.28** Alternative representations of interslice forces for a slope with internal reinforcing elements; (*a*) slope with internal reinforcing elements; (*b*) reinforcement and soil + water forces combined; (*c*) reinforcement and soil + water forces considered separately.

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for any point along the reinforcement and to be able to compute the force wherever the reinforcement crosses a boundary. There is very little difference between the scheme used to compute the force where the reinforcement crosses the slip surface and the scheme used to compute the force where the reinforcement crosses a slice boundary. Therefore, once a suitable computational scheme has been developed, it can be used to compute the reinforcement force at points where the reinforcement crosses both the interslice boundaries and the slip surface. Unlike calculating the water pressure force on an interslice boundary, it is relatively easy to compute the reinforcement force on an interslice boundary. Thus, it is also relatively easy to consider the reinforcement force separately from the remaining interslice forces. Also, the reinforcement forces may act in directions that are very different from the interslice forces in the soil and water. By treating the reinforcement forces separately from the forces due to the soil and water, more rational assumptions can be made about the inclinations of the interslice forces.

Separation of the reinforcement force from the force due to the soil and water at the interslice boundaries also produces a more realistic set of internal forces and usually a more stable numerical solution to the equilibrium equations. If the reinforcement forces are applied where the reinforcement intersects both the slip surface and each interslice boundary, the forces that are applied to each slice will be more realistic. For example, for a slice like the one shown in Figure 6.29*b*, the actual force exerted on the slice by the reinforcement will be equal to the difference between the force where the reinforcing element enters the slice at the left interslice boundary and exits the slice (slip surface) at the bottom. The net reinforcement force, $T_{i+1} - T_i$, on the slice in this case may be quite small. In contrast, if only the reinforcement force, $T_{i+1}$, acting on the base of the slice is applied to the slice, the applied reinforcement force could be quite large. For equilibrium to be satisfied with the reinforcement force applied only to the base of the slice, the unknown interslice force on the left of the slice might need to be much larger than the unknown interslice force on the right of the slice, and the inclination of the interslice forces on the left and right of the slice might be very different. Such abrupt changes in the magnitude and inclination of the interslice forces can lead to numerical problems in the solution of the equilibrium equations.

The reason for separating the interslice forces due to reinforcement from those in the soil can be further seen by considering a slice such as the one shown in Figure 6.29*c* where the reinforcing element passes entirely across the slice, intersecting both vertical boundaries. The actual force exerted on the slice by the reinforcement will be the force that is transferred by shear (load transfer) between the reinforcing element and the soil as the reinforcing element passes through the slice. This force is properly represented as the difference between the forces in the reinforcing element at the two sides of the slice (i.e., $T_{i+1} - T_i$). If the reinforcement forces on each side of the slice ($T_i$ and $T_{i+1}$) are computed and applied as known forces, separately from the unknown interslice forces in the soil, the slice will be assured of receiving the proper contribution of the reinforcement.

## COMPUTATIONS WITH ANISOTROPIC SHEAR STRENGTHS

Earlier, the shear strength has been expressed by appropriate values of cohesion ($c$, $c'$) and friction angle ($\phi$, $\phi'$). If the shear strengths are anisotropic, the values of cohesion and friction angle may depend on the orientation of the failure plane. Typically, the shear strength in such cases is the undrained shear strength and $\phi$ will be equal to zero, so the anisotropic shear strength is expressed by the variation in undrained shear strength, $s_u$, with orientation of the failure plane. To perform stability computations when the strengths vary with the failure plane orientation, it is simply necessary to assign appropriate values of cohesion and friction to each slice based on the inclination of the bottom of the slice (slip surface). Once the strengths are assigned, computations proceed in the normal manner.

## COMPUTATIONS WITH CURVED FAILURE ENVELOPES

All of the equations presented previously assume that the shear strength will be represented by a linear Mohr–Coulomb failure envelope defined by an intercept (cohesion) and a slope (friction angle). However, failure envelopes are often curved. Curved failure envelopes require somewhat more effort in a stability analysis. If the Mohr failure envelope is curved, the shear strength varies with the normal stress ($\sigma$, $\sigma'$), and the normal stress must be known before the shear strength is known. Only the Ordinary Method of Slices allows the normal stress to be computed without knowing the shear strength; all other procedures require that the shear strength be known before the normal stress

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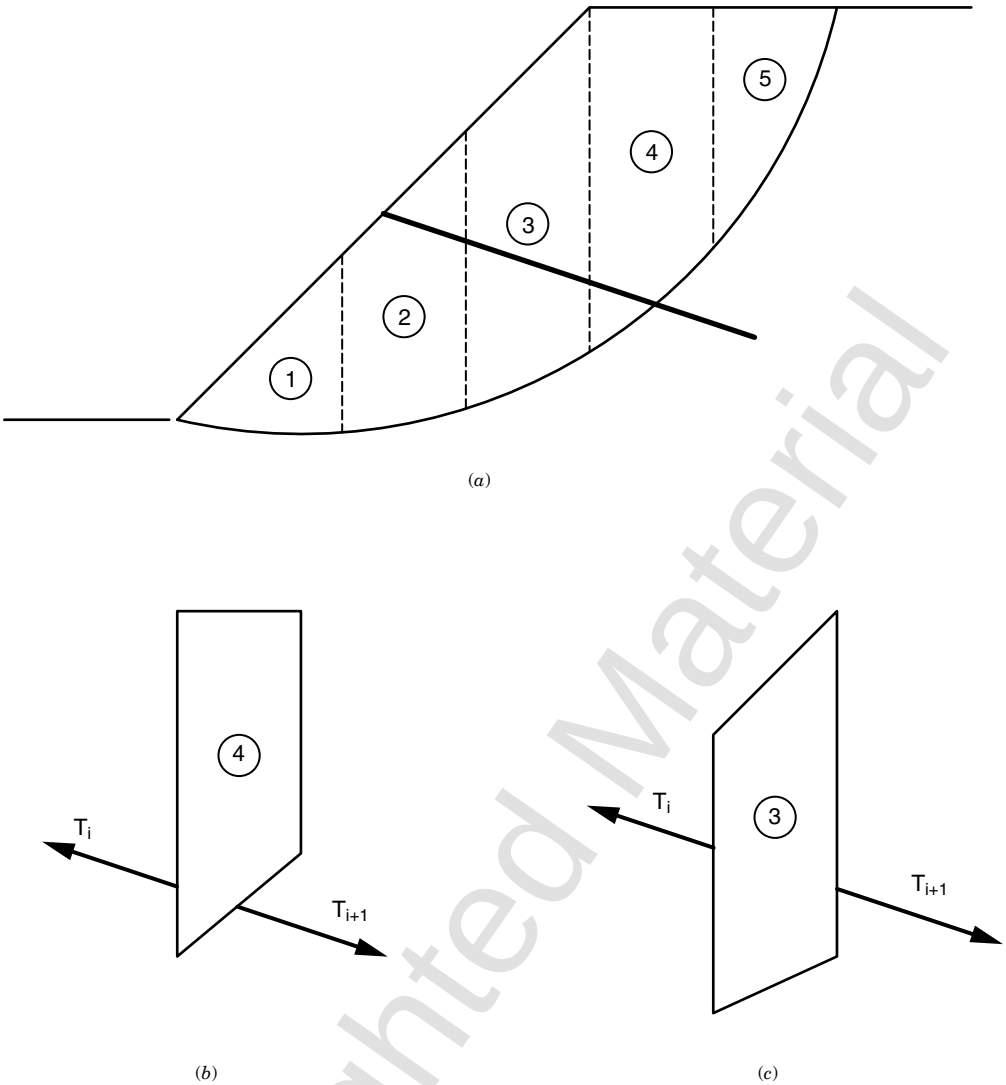



**Figure 6.29** Reinforcement forces acting on individual slices: (*a*) slope with internal reinforcing elements; (*b*) reinforcement forces for slice 4; (*c*) reinforcement forces for slice 3.

can be computed (solved for). In these procedures it is necessary to estimate the shear strength first and then perform the necessary stability computations to calculate the normal stress (and factor of safety). Once the normal stress is found, new shear strengths can be estimated and the process repeated until convergence is reached. Shear strengths may be assigned as a pure cohesion ($\phi = 0$), based on the estimated normal stress (Figure 6.30*a*). Alternatively, the shear strength may be represented by a cohesion and a friction angle representing a failure envelope tangent to the curved envelope at the estimated normal stress (Figure 6.30*b*). The second approach of using an envelope tangent to the curved envelope is more complex but produces faster convergence in the necessary trial-and-error procedures. However, both methods produce identical results once convergence is reached.

## ALTERNATIVE DEFINITIONS OF THE FACTOR OF SAFETY

Up to this point the factor of safety has been defined with respect to the shear strength of the soil. This definition is the one generally used for slope stability analyses, and this definition [Eq. (6.1)] is recommended and used throughout the book unless otherwise noted. However, other definitions for the factor of

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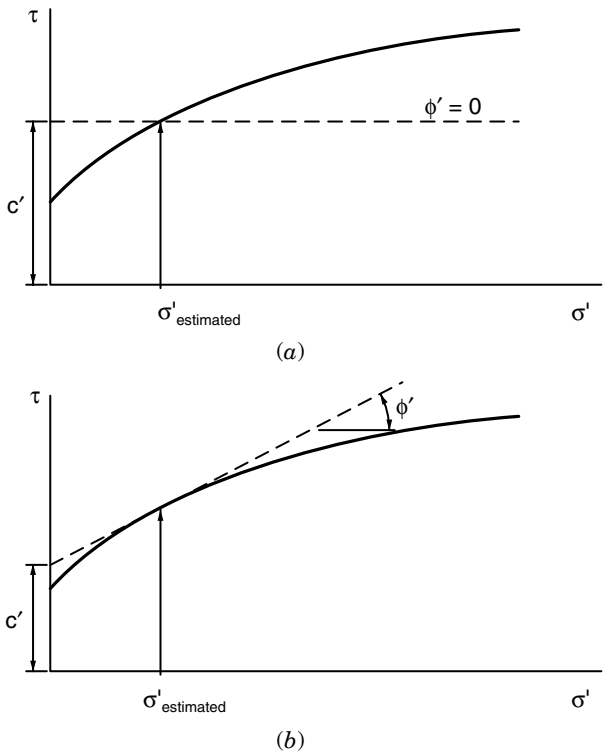



**Figure 6.30** Alternative representations of nonlinear Mohr failure envelopes by equivalent values of cohesion and friction angle: (*a*) representation by equivalent strength $c$ ($\phi = 0$); (*b*) representation by equivalent tangent values of cohesion and friction.

safety have sometimes been used. These usually lead to different values for the factor of safety. Two such definitions for the factor of safety are discussed below.

### Factor of Safety for Load

One definition of the factor of safety is with respect to load. The case where the factor of safety is usually defined with respect to load is bearing capacity. The factor of safety for bearing capacity defined in terms of load is

$$F = \frac{\text{load required to cause failure}}{\text{actual applied load}} \tag{6.94}$$

To illustrate the difference between this definition of the factor of safety and the factor of safety defined with respect to shear strength, consider the footing shown in Figure 6.31. The footing is 8 ft wide, rests on cohesionless soil, and exerts a bearing pressure of 10,000 psf on the soil. The ultimate bearing pressure, $q_{ult}$, required to cause failure of the footing is expressed as

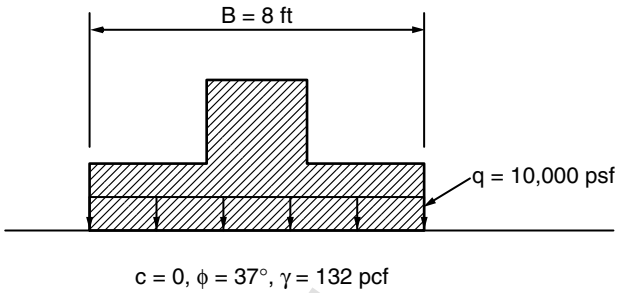



**Figure 6.31** Footing problem used to illustrate differences between factors of safety applied to load and to soil shear strength.

$$q_{ult} = \tfrac{1}{2}\gamma B N_\gamma \tag{6.95}$$

where $N_\gamma$ is a bearing capacity factor, which depends only on the angle of internal friction, $\phi$. For a friction angle of 37° the value of $N_\gamma$ is 53 and the ultimate bearing capacity is then

$$q_{ult} = \tfrac{1}{2}\gamma B N_\gamma = \tfrac{1}{2}(132)(8)(53) = 27{,}984 \approx 28{,}000 \text{ lb} \tag{6.96}$$

Thus, the factor of safety for load is

$$F = \frac{28{,}000}{10{,}000} = 2.80 \tag{6.97}$$

Equation (6.96) is actually a limit equilibrium equation that relates the shear strength ($\phi$) to the bearing pressure that produces equilibrium when the shear strength of the soil is fully developed. If, instead, we consider some fraction of the shear strength being developed such that

$$\tan \phi_d = \frac{\tan \phi}{F} \tag{6.98}$$

where $F$ is the factor of safety on shear strength and $\phi_d$ is the developed friction angle, we can write

$$q_{equil} = \tfrac{1}{2}\gamma B N_{\gamma\text{-developed}} \tag{6.99}$$

where $q_{equil}$ is the equilibrium bearing pressure and $N_{\gamma\text{-developed}}$ is the value of $N_\gamma$ based on the developed friction angle, $\phi_d$. If we let the applied bearing pressure of 10,000 psf shown in Figure 6.31 be the equilibrium pressure, we can find the value of the factor of safety with respect to shear strength (and $\phi_d$) that satisfies equilibrium [Eq. (6.99)]. This was done by trial and error, assuming a factor of safety and computing the

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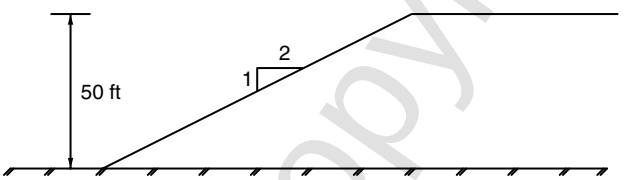
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corresponding equilibrium bearing pressure. The results are summarized in Table 6.5. Referring to Table 6.5, it can be seen that a factor of safety of 1.25 applied to the shear strength produces equilibrium of the footing under the applied load of 10,000 psf. This value (1.25) for the factor of safety applied to shear strength is considerably less than the corresponding value of 2.80 applied to load, which was calculated earlier [Eq. (6.97)]. Factors of safety that are applied to load for bearing capacity are thus not comparable to the factors of safety applied to shear strength, as used for slope stability analyses.

The factor of safety with respect to shear strength is closer to 1.0 than the factor of safety with respect to load; however, the magnitude of the difference varies significantly depending on the value of $\phi$. Consider, for example, the slope shown in Figure 6.32. Factors of safety were calculated for the three different sets of shear strength parameters shown in this figure. The values of the shear strength parameters were selected so that the factor of safety with respect to shear strength [Eq. (6.1)] was approximately 1.5. For each slope the factors of safety both with respect to shear strength and with respect to load were calculated. The factor of safety with respect to load was calculated by multiplying the unit weight of the soil by a factor of safety until the slope was in just-stable equilibrium with the shear strength fully developed. The factors of safety for shear strength and load are summarized for the three slopes in Table 6.6. For the first set of shear strengths ($\phi = 0$) the factors of safety for shear strength and load are identical. This will always be the case when $\phi = 0$, for slope stability as well as for bearing capacity problems such as the footing shown previously. For the second set of shear strength parameters, the factor of safety with respect to load was approximately 11, whereas the factor of safety for shear strength was approximately 1.5. This represents a difference of over 700%. Finally, for the third set of shear strength parameters, $\phi = 36.9°$ and $c = 0$, the factor of safety with respect to shear strength was 1.5, whereas the factor of safety with respect to load is infinite (i.e., no matter how large the weight of soil, the shear strength always remains greater than the shear stress). In summary, the factors of safety for shear strength and for load can vary from being the same to being very different for large values of $\phi$, and the two values are not comparable. Because the soil shear strength is one of the largest unknowns in a slope stability analysis—certainly it presents greater uncertainty than the unit weight of soil in almost all instances—it seems logical to apply the factor of safety to shear strength.

**Table 6.5 Summary of Computed Equilibrium Bearing Pressures for Various Factors of Safety on Shear Strength**

| Assumed $F$ | $\phi_d$ (deg) | $N_{\gamma\text{-developed}}$ | $q_{equil}$ (psf) |
|---|---|---|---|
| 1.00 | 37.0 | 53 | 28,000 |
| 1.25 | 31.1 | 19 | 10,000 |
| 1.50 | 26.7 | 9 | 4,800 |




| Property Set | c (psf) | ϕ (degrees) | γ (pcf) |
|---|---|---|---|
| 1 | 1065 | 0 | 115 |
| 2 | 285 | 23.7 | 128 |
| 3 | 0 | 36.9 | 125 |

**Figure 6.32** Slope and soil properties used to compute factors of safety applied to shear strength and applied to load.

### Factor of Safety for Moments

Another definition that has been suggested for the factor of safety is one based on moments. In this case the factor of safety is defined as the ratio of the available resisting moment divided by the actual driving moment:

$$F = \frac{\text{available resisting moment}}{\text{actual driving moment}} = \frac{M_r}{M_d} \quad (6.100)$$

This can be rearranged and written as

**Table 6.6 Sumary of Computed Factors of Safety Applied to Shear Strength and Applied to Load for Example Slope and Three Sets of Soil Properties**

| | Factor of safety applied to: | |
|---|---|---|
| Property set | Shear strength | Load |
| 1 ($\phi = 0$) | 1.50 | 1.50 |
| 2 ($c > 0$, $\phi > 0$) | 1.50 | 11.00 |
| 3 ($c = 0$) | 1.50 | Infinite |

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$$M_d - \frac{M_r}{F} = 0 \tag{6.101}$$

Equation (6.101) is an equilibrium equation that expresses a balance between the driving moment and a developed resisting moment that is equal to the total available resisting moment factored by the factor of safety. If the resisting moment is due entirely to the shear strength of the soil, the factor of safety applied to the resisting moment is the same as the factor of safety defined earlier with respect to shear strength. In fact, it was shown earlier for the Swedish Circle method [Eq. (6.28)] that the factor of safety defined with respect to shear strength was equal to the ratio of moments expressed by Eq. (6.100). In this case there is no difference between the factors of safety defined with respect to shear strength [Eq. (6.1)] and with respect to moments [Eq. (6.100)].

If instead of a simple slope, where all resistance is from the shear strength of the soil, there are additional forces due to reinforcement, the two definitions of factor of safety [Eqs. (6.1) and (6.100)] can be quite different. Also, the definition of the factor of safety as a ratio of moments can be ambiguous. To illustrate this, consider the slope and circular slip surface shown in Figure 6.33. This slope has a single layer of reinforcement. Let the moment taken about the center of the circle due to the reinforcement be designated as $M_t$. Let the corresponding moments due to the available shear strength be designated as $M_s$, and the moment due to the weight of soil be designated as $M_d$. We can now define a factor of safety with respect to resisting and driving moments. If we choose to add the moment due to the reinforcement to the resisting moment due to the shear strength of the soil, we can write

$$F = \frac{M_s + M_t}{M_d} \tag{6.102}$$

Alternatively, we can choose to subtract the restoring moment due to the reinforcement from the driving moment due to the soil weight and write

$$F = \frac{M_s}{M_d - M_t} \tag{6.103}$$

Equations (6.102) and (6.103) both represent legitimate definitions for the factor of safety defined as a ratio of moments; however, the two definitions give different values for the factor of safety. Equations (6.102) and (6.103) are more easily interpreted if we rewrite them as follows: For Eq. (6.102) we can write

$$M_d = \frac{M_s}{F} + \frac{M_t}{F} \tag{6.104}$$

and for Eq. (6.103) we can write

$$M_d = \frac{M_s}{F} + M_t \tag{6.105}$$

Both of these equations can be interpreted as equilibrium equations. The first equation, where the contribution of the reinforcement was added to the resisting moment, states that the driving moment is balanced by moments due to the developed shear strength and the developed reinforcement forces, where developed values are the available values reduced by a factor of safety, $F$. Thus, the factor of safety in Eq. (6.104) is applied equally to the reinforcement forces and the shear strength. The second of the two equilibrium equations [Eq. (6.105)], where the reinforcement contribution was used to reduce the driving moments, states that the driving moment is in equilibrium with the full reinforcement force, plus the factored resistance due to the shear strength of the soil. In this case the factor of safety is applied only to the soil shear strength.

To illustrate the differences in values computed for the factors of safety of reinforced slopes depending on how the factor of safety is defined, consider the slope

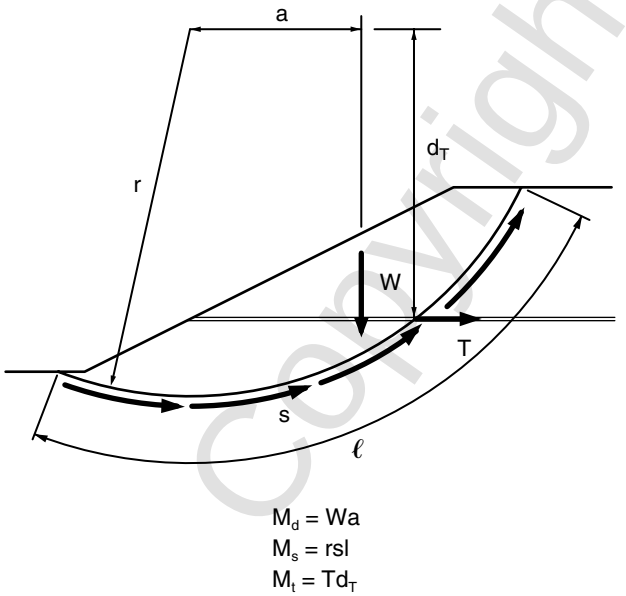



**Figure 6.33** Simple reinforced slope with driving and resisting moments.

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shown in Figure 6.34. This slope has a single layer of reinforcement and $\phi = 0$. The factor of safety for the unreinforced slope is 0.91, thus indicating that the reinforcement is necessary to make the slope stable. Factors of safety were first computed by applying the factor of safety to shear strength only [Eq. (6.105)] and to both shear strength and reinforcement force equally [Eq. (6.104)]. A third factor of safety was computed by applying the factor of safety to only the reinforcement force (i.e., the shear strength was assumed to be fully mobilized and the reinforcement force was reduced by the factor of safety). The three different values for the factor of safety shown in Figure 6.34 range from approximately 1.3 to 4.8, a difference of over threefold. Clearly, the manner in which the factor of safety is defined will affect the computed value.

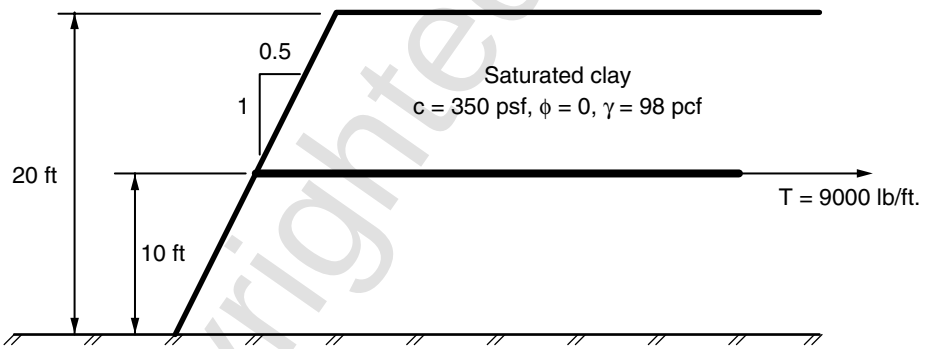
Although any of the foregoing definitions for factor of safety could be used to compute a factor of safety, only Eq. (6.105) is consistent with the definition of factor of safety generally used for slope stability analyses throughout this book. Instead of defining and computing a factor of safety that is applied equally to the reinforcement forces and soil strength [Eq. (6.104)] or to only the reinforcement force, it seems more appropriate first to apply a suitable factor of safety to the reinforcement forces before any slope stability computations begin and then compute a separate factor of safety with respect to the shear strength of the soil. This approach is recommended and is discussed further in Chapter 8.

## PORE WATER PRESSURE REPRESENTATION

Whenever the shear strength of one or more materials is expressed in terms of effective stresses, the pore water pressures must be determined and represented in the slope stability analysis. Several methods exist for doing this, depending on the seepage and groundwater conditions and the degree of rigor required. The various methods are described and discussed in this section.

### Flow Net Solutions

When steady-state seepage conditions exist in a slope, a graphical flow net solution can be used to determine the pore water pressures. For most slopes this requires determining the location of a *line of seepage* representing the uppermost flow line, and then constructing families of curves representing flow and equipotential




| Case | Factor of Safety |
|---|---|
| Factor of safety applied to shear strength only. | 1.51 |
| Factor of safety applied to reinforcement force only. | 4.82 |
| Factor of safety applied to both shear strength and reinforcement force. | 1.32 |

**Figure 6.34** Reinforced slope with computed factors of safety defined (applied) in three different ways.

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lines in the region of saturated flow, below the line of seepage (Casagrande, 1937). Once a correct flow net has been constructed, pore water pressures can be calculated at any point desired; pore water pressures are usually assumed to be zero above the line of seepage.

Although flow nets provide an accurate representation of pore water pressures, flow nets are difficult and tedious to use in slope stability computations. For each slice of each trial slip surface, pore water pressures must be calculated. This generally requires some interpolation between the equipotential lines in the flow net to get the pore water pressures at each slice. If done manually, the task is extremely tedious. It is also difficult to use a flow net in computer calculations. To do so, the pore water pressures must first be computed from the flow net at selected points. Usually, points corresponding to the intersections of the flow lines and equipotential lines are used. Once the pressures are computed, which must be done manually, the digitized values of pressure need to be entered into a computer program. In the computer program it is necessary to interpolate the values of pressure that were input to get pore water pressures at the center of the base of each slice. Although the interpolation process can be automated, much as it is when using pore water pressures calculated from finite element analyses, a substantial effort is still required to construct a flow net and then compute the pore water pressures for entry into the computer program. Also, for complex slopes it is impractical to construct a flow net by hand.

### Numerical Solutions

Today, most analyses of seepage and groundwater flow, for any but the simplest conditions, are conducted using finite difference or finite element numerical solutions. Because of the great flexibility that it provides, most such analyses are done using the finite element method. Results of such analyses consist of values of pore water pressure at each of a number of nodal points in the finite element mesh.

Most of the earlier and even some of today's finite element modeling schemes model only the region of saturated flow, below the line of seepage. Essentially, these schemes mimic what is done with flow net solutions by establishing a line of seepage and assuming no flow above the line of seepage. Various schemes have been used to determine the location of the line of seepage; including adjusting the geometry of the finite-mesh and truncating the finite element mesh at the point where it intersects the line of seepage (zero pressure line). With these schemes pore water pressure are calculated only in the region of saturated flow, where the pore water pressures are positive. Pore water pressures are assumed to be zero above the line of seepage.

Most current finite element modeling schemes model the entire cross section of a slope, including the region where the pore water pressures are negative and the soil may be unsaturated. These schemes employ finite elements everywhere there is soil, and the hydraulic conductivity is adjusted to reflect the pore water pressures and degree of saturation. Both positive and negative values of pore water pressure are calculated. However, for slope stability analyses the pore water pressures are usually assumed to be zero in the region where negative pressures have been calculated.

Regardless of the finite element scheme used, results of a finite element analysis consist of the value of pore water pressure at each nodal point. These values must then be used along with a suitable interpolation scheme to calculate the pore water pressures at the center of the base of individual slices along a slip surface.

### Interpolation Schemes

Several interpolation schemes have been used to calculate the pore water pressures along the slip surface from the results of finite element analyses. Several of these schemes are described below.

**Three- and four-point interpolation.** One of the earliest schemes used to interpolate pore water pressures from gridded data for slope stability calculations was based on a three- or four-point interpolation function (Wright, 1974; Chugh, 1981). In this scheme the three or four points where pore water pressures were defined (e.g., nodal points) that are closest to the point where pore water pressures are to be calculated are located. If four points are used, the pore water pressures are then interpolated using an equation of the form

$$u = a_1 + a_2x + a_3y + a_4xy \qquad (6.106)$$

where $x$ and $y$ are coordinates and $a_1$, $a_2$, $a_3$, and $a_4$ are four coefficients that are evaluated using the coordinates and pore water pressures at the four interpolation points (nodal points). Once the coefficients are determined, Eq. (6.106) is used to compute the pore water pressure at the point on the slip surface. If three points are used for interpolation, the form of the equation is

$$u = a_1 + a_2x + a_3y \qquad (6.107)$$

but otherwise, the procedure is the same as for four points. This three- or four-point interpolation scheme has problems that sometimes lead to erroneous values,

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especially when pore water pressures are interpolated (actually, extrapolated) outside the perimeter of the region formed by the three or four interpolation points. Some improvement was achieved in this scheme through an ''averaging'' scheme proposed by Chugh (1981); however, the scheme still can lead to erroneous values, especially when three or more of the interpolation points lie along a nearly straight line.

**Spline interpolation.** Two-dimensional interpolation schemes based on spline surfaces are more rigorous and overcome some of the limitations of the three- and four-point schemes described above. In the spline interpolation schemes a much larger number of points is used to interpolate the pore water pressures at any given point (Geo-Slope, 2002; RocScience, 2002). Also, a much larger system of equations must be solved for the interpolation coefficients. The procedures work well but can be relatively time consuming in terms of computational effort and computer memory requirements. These schemes may also result in values being extrapolated outside the actual range of the interpolation point data (i.e., the pore water pressures may exceed or be less than the respective highest and lowest values at points used for interpolation).

**Finite element shape functions.** Another scheme that should be quite accurate, but has not been widely used, employs the same shape functions that are used for the finite element formulation to interpolate pore water pressures once the finite element solution is completed. This scheme has been used by the writers as well as by Geo-Slope (2002). The scheme closely integrates the finite element solution with the subsequent interpolation of pore water pressures and thus should not introduce additional errors resulting from the interpolation scheme. With this scheme the element that contains the point where pore water pressures are to be interpolated is found first. The pore water pressures are then calculated using the values of the head at the adjoining nodal points and the finite element shape (interpolation) functions. This scheme is relatively complex computationally, but more important, the scheme is only well suited for interpolation of pore water pressures when the pore water pressures have been computed using the finite element method. The scheme requires more close integration of the finite element and slope stability analyses than most other schemes.

**Triangulated irregular network scheme.** One of the best interpolation schemes is based on use of a triangulated irregular network (TIN) for interpolation (Wright, 2002). The TIN consists of a set of triangles whose vertices coincide with the points where pore water pressures are defined and which cover without overlapping the entire region where pore water pressures are defined. A convenient scheme for creating the triangles is the *Delaunay triangulation scheme* (Figure 6.35). In this scheme the triangles are created such that no interpolation point lies within the circumcircle of any triangle (Lee and Schachter, 1980; Watson and Philip, 1984). A *circumcircle* is the circle passing through the three vertices of a triangle and contains the three vertices, but there are no points inside the circle.[11]. Robust algorithms exist for creating a Delaunay triangulation for any series of discrete points.

Once a Delaunay triangulation is created interpolation consists of a two-step process for interpolating pressures at any point: First, the triangle that contains the point where pressures are to be interpolated is located. Then, the pore water pressures are interpolated linearly from the values of pore water pressure at the three vertices of the enclosing triangle. The interpolation equation is of the form of Eq. (6.107), but differs from the previous scheme in that the interpolation point never lies outside the perimeter of the triangle formed by the three points used for interpolation. Also by using the Delaunay triangulation scheme it is usually possible to avoid having the three points used for interpolation being located on essentially a straight line. One of the most important advantages of the TIN-based scheme is that algorithms exist for very quickly locating the appropriate triangle that contains the point where pressures are to be interpolated, and thus, the points to be used for interpolation are located quickly (Lee and Schachter, 1980; Mirante and Weingarten, 1982; Jones, 1990). The three- and four-point schemes described earlier as well as schemes using the finite element shape functions may involve a significant amount of time being spent on locating the appropriate points or finite element that are to be used for the interpolation. The TIN-based scheme also has the advantage of using a very simple linear interpolation function [Eq. (6.107)] which requires very little computational time and computer memory for storage.

Another advantage of the TIN-based interpolation scheme is that it is applicable to interpolation using any type of irregularly gridded data, not just the results from finite element analyses. For example, there may be cases where some pore water pressures are recorded in piezometers or by groundwater observations. These measured data may then be supplemented by additional data points based on judgment and interpretation to provide a grid of values. Such values can then easily

[11] It is possible for more than three points to lie on a circumcircle (e.g., if four points form a rectangle); however, no point will ever lie inside the circumcircles in a Delaunay triangulation.

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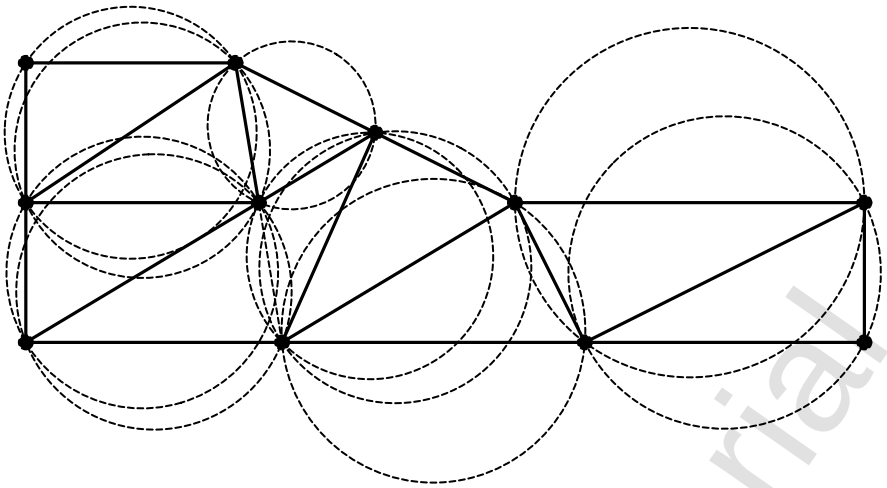


**Figure 6.35** Delaunay triangulation of a series of interpolation points (e.g., nodal points) with accompanying circumcircles.

be used to compute values of pore water pressure at other points using the TIN-based interpolation scheme.

A related advantage of TIN-based interpolation schemes is that once such schemes are developed and implemented in computer codes, they can also be used for other forms of interpolation required in slope stability computations. For example, TIN-based schemes are useful for interpolating values of undrained shear strength. This is particularly useful when the undrained shear strengths vary both vertically and horizontally. Typical examples where such spatial variation in undrained shear strength occur include some mine tailings disposal structures and clay foundations beneath embankments built with staged construction techniques.

### Phreatic Surface

Flow net and finite element seepage solutions are complex and time consuming to perform. In many cases it is more appropriate to use simple approximations of the seepage and pore water pressures in a slope. Considering that groundwater and seepage conditions are often not well known, the simple approximations may be more than justified. One approximation is to define the pore water pressures using a line that represents a phreatic surface. The phreatic surface corresponds to the line of seepage from a flow net [i.e., the phreatic surface is considered to be a flow line and a line of zero (atmospheric) pressure]. The line of zero pressure from a finite element solution is also often considered equivalent to the phreatic surface even though in this case the zero pressure line may not be a flow line.

When the pore water pressures are defined by a phreatic surface, the pore water pressure is equal to the product of the pressure head, $h_p$, and the unit weight of water, $\gamma_w$:

$$u = h_p \gamma_w \tag{6.108}$$

If the phreatic surface is a straight line and the equipotential lines are also straight lines (Figure 6.36*a*), perpendicular to the phreatic surface, the pressure head is related to the vertical depth, $z_p$, below the phreatic surface by

$$h_p = z_p \cos^2\beta \tag{6.109}$$

where $\beta$ is the slope of the phreatic surface. If, instead, the phreatic surface and equipotential lines are curved as shown in Figure 6.36*b*, the pore water pressure can be expressed by the following inequalities:

$$z_p \gamma_w \cos^2\beta' < u < z_p \gamma_w \cos^2\beta'' \tag{6.110}$$

where $\beta'$ is the slope of the phreatic surface at the point where the equipotential line intersects the phreatic surface and $\beta''$ is the slope of the phreatic surface directly above the point of interest. In this case ($\beta'' < \beta'$) it is conservative to express the pore water pressure as

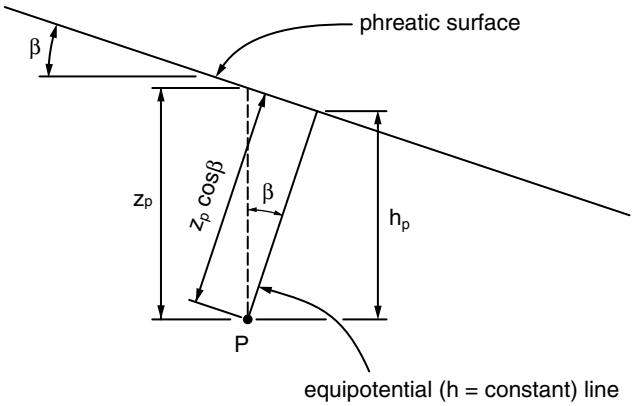
$$u = z_p \gamma_w \cos^2\beta'' \tag{6.111}$$

where again $\beta''$ is the slope of the phreatic surface directly above the point of interest.

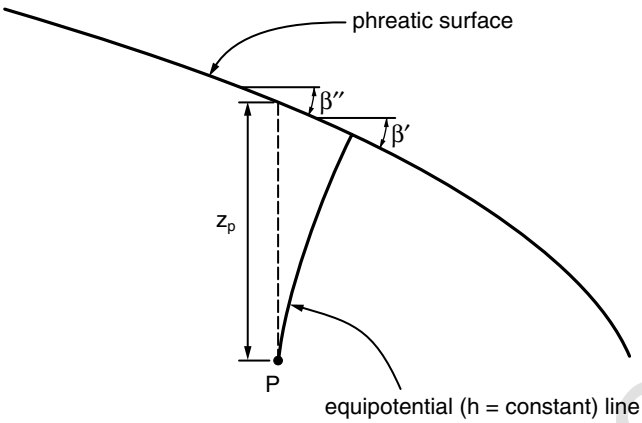
Several computer programs have used Eq. (6.111) or a similar form to approximate the pore water pressures from a phreatic surface. This seems to provide a

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(*a*)




(*b*)

**Figure 6.36** (*a*) Linear and (*b*) curved phreatic surfaces used to approximate pore water pressures.

reasonable approximation for many cases, but as shown later the approximation can also lead to unconservative results depending on seepage conditions.

### Piezometric Line

As a further approximation and simplification of pore water pressures a piezometric line may be used. With a piezometric line the pore water pressures are computed by multiplying the vertical depth below the piezometric line by the unit weight of water. Thus,

$$u = z_p \gamma_w \tag{6.112}$$

This representation is sometimes considered to be a conservative representation of the pore water pressures compared to the phreatic surface described previously. However, the differences between the two representations of pore water pressure are typically small because the slope ($\beta$) for most surfaces is small. Furthermore, because the phreatic surface is at times unconservative (i.e., it can underestimate the pore water pressures), representation by a piezometric line may be just as suitable as a phreatic surface. It should be recognized that both represent approximations that may or may not be valid, depending on the particular seepage conditions in a slope.

### Examples

To illustrate differences in the various representations of pore water pressure described above, analyses of two examples are presented. For each of these two examples a finite element steady-state seepage analysis was performed first using the GMS/SEEP2D software (EMRL, 2001) to calculate the pore water pressures. The seepage analyses were performed by modeling the entire soil cross section shown for each example and using appropriate values of saturated or unsaturated hydraulic conductivity depending on the pore water pressures. Pore water pressures calculated from the finite element seepage analyses were then used to interpolate pore water pressures along the slip surface for each trail slip surface and slice. Pore water pressures were interpolated using the TIN-based interpolation scheme described earlier. A phreatic surface was also established by locating the line (contour) corresponding to zero pore water pressure. Pore water pressures were calculated from the phreatic surface using Eq. (6.111). Finally, a piezometric line was defined from the line of zero pressure (i.e., the phreatic surface and piezometric line were the same). Pore water pressures were calculated from the piezometric line using Eq. (6.112).

Factors of safety were calculated for both examples using each of the pore water pressure representations described above. For each slope and representation of pore water pressure the critical circle with the minimum factor of safety was found.

**Example 1.** The slope for the first example is shown in Figure 6.37. The slope and foundation are homogeneous and composed of the same soil. Total heads, expressed relative to a datum at the bottom of the foundation, are 75 ft at the entrance to the cross section and 40 ft at the ground surface beyond the toe of the slope. The slope face is assumed to be a free discharge surface, and it is assumed that no infiltration occurs along the slope face or behind the crest of the slope.

The contour of zero pressure determined from the finite element seepage analysis is shown in Figure 6.38. In the finite element analysis, pore water pressures were negative above and positive below this line. In the subsequent slope stability analyses, the zero

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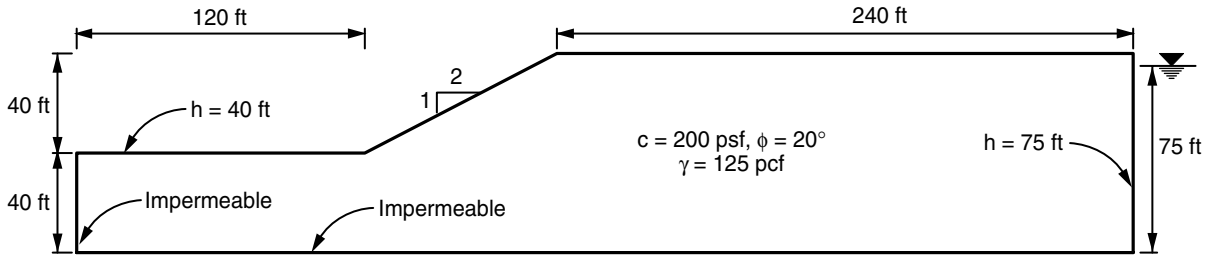



**Figure 6.37** Homogeneous slope used to illustrate different schemes for representing pore water pressures in slope stability analyses.

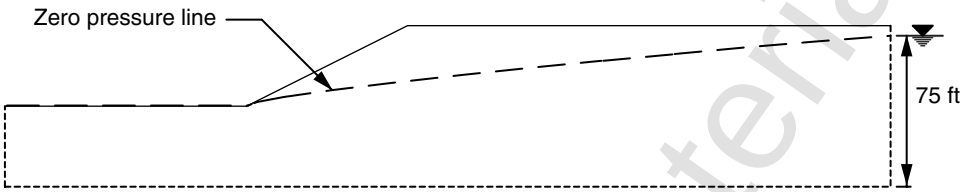



**Figure 6.38** Zero-pressure line (contour) determined using finite element analysis, used to represent the phreatic surface and the piezometric line for a homogeneous slope.

pressure line was used to represent both the piezometric line and the phreatic surface.

Factors of safety calculated for each of the three representations of pore water pressure are summarized in Table 6.7. The three values shown for the different representations of pore water pressures are all extremely close; the values are shown to three decimal places to illustrate the differences. All three representations produced approximately the same factor of safety of approximately 1.14 to 1.15. The phreatic surface in this case produced a slightly higher factor of safety because of a slight upward component of flow beyond the toe of the slope, which causes the phreatic surface and Eq. (6.111) to underestimate pore water pressures slightly.

**Table 6.7 Summary of Computed Factors of Safety for Three Different Representations of Pore Water Pressure: Homogeneous Slope and Foundation**

| Pore water pressure representation | Factor of safety |
|---|---|
| Finite element analysis with pore pressures interpolated from nodal points values using triangle-based interpolation scheme | 1.138 |
| Phreatic surface approximation | 1.147 |
| Piezometric line approximation | 1.141 |

**Example 2.** The slope for the second example consists of the earth embankment dam resting on a layered soil foundation as shown in Figure 6.39. Soil properties are shown in Figure 6.39. The foundation consists of a low-permeability clay layer underlain by a more permeable layer of sand. A significant amount of underseepage occurs through the more permeable sand layer at depth and produces upward flow near the downstream portion of the dam. The zero-pressure line determined from the finite element analyses is shown in Figure 6.40. This line was used as a phreatic surface and piezometric line for the slope stability analyses.

Although the minimum factor of safety for this slope occurs for very shallow circles (sloughs) near the toe of the slope, deeper slip surfaces are likely to be of interest as well. Therefore, the overall minimum factor of safety (shallow circles) and the minimum factor of safety for circles tangent to elevation 197 (bottom of clay) were both calculated. The factors of safety are summarized in Table 6.8. Factors of safety calculated by both the phreatic surface and piezometric line representations of pore water pressures were essentially the same because of the relatively flat zero-pressure surface. However, both the piezometric line and phreatic surface representations of pore water pressures produced factors of safety that ranged from 14 to 19% greater than the factors of safety calculated from the finite element solution and interpolation. These relatively large differences are due to the upward flow of water through the clay foundation, which is not represented well by either a phreatic surface or a single piezometric line.

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| Material | k - ft/min | c′ - psf | ϕ′ - degrees | Unit wt. - pcf |
|---|---|---|---|---|
| Outer shell | $1 \times 10^{-2}$ | 0 | 34 | 125 |
| Clay core | $1 \times 10^{-6}$ | 100 | 26 | 122 |
| Foundation clay | $1 \times 10^{-5}$ | 0 | 24 | 123 |
| Foundation sand | $1 \times 10^{-3}$ | 0 | 32 | 127 |

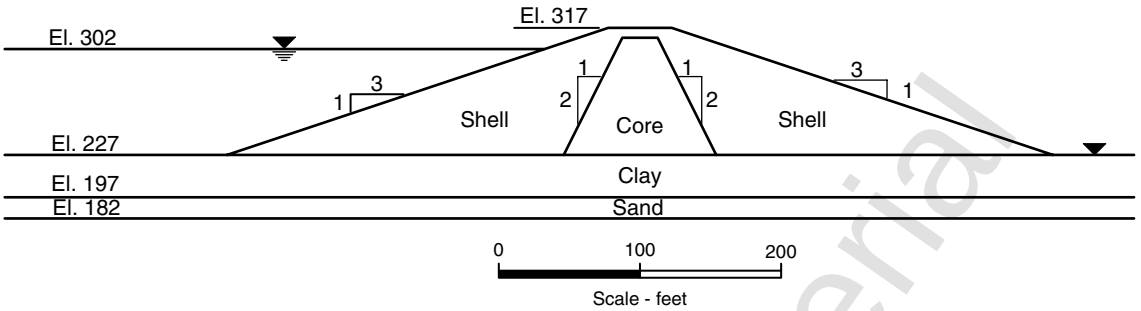



**Figure 6.39** Embankment dam on layered foundation used to illustrate different schemes for representing pore water pressures in slope stability analyses.

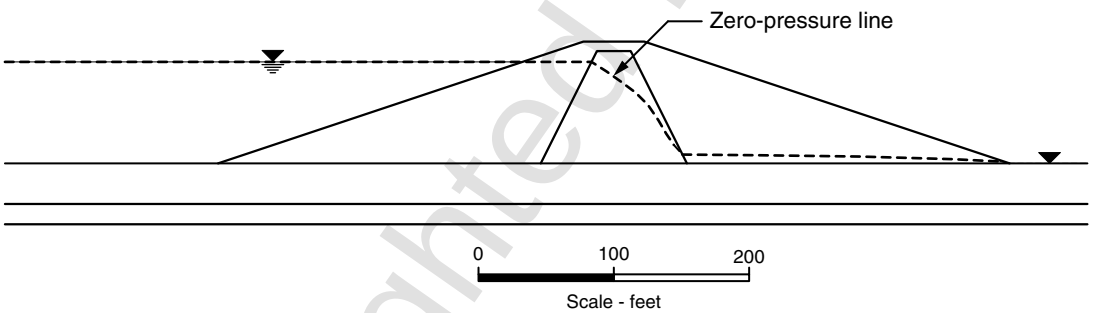



**Figure 6.40** Zero-pressure line (contour) determined using finite element analysis, used to represent the phreatic surface and the piezometric line for an embankment dam.

**Table 6.8 Summary of Computed Factors of Safety for Three Different Representations of Pore Water Pressure: Embankment Dam on Layered Soil Foundation**

| Pore water pressure representation | Minimum factors of safety | |
|---|---|---|
| | Overall critical circle | Critical circle tangent to elev. 197 |
| Finite element analysis with pore pressures interpolated from nodal points values using triangle-based interpolation scheme | 1.11 | 1.37 |
| Phreatic surface approximation | 1.32 | 1.57 |
| Piezometric line approximation | 1.30 | 1.57 |

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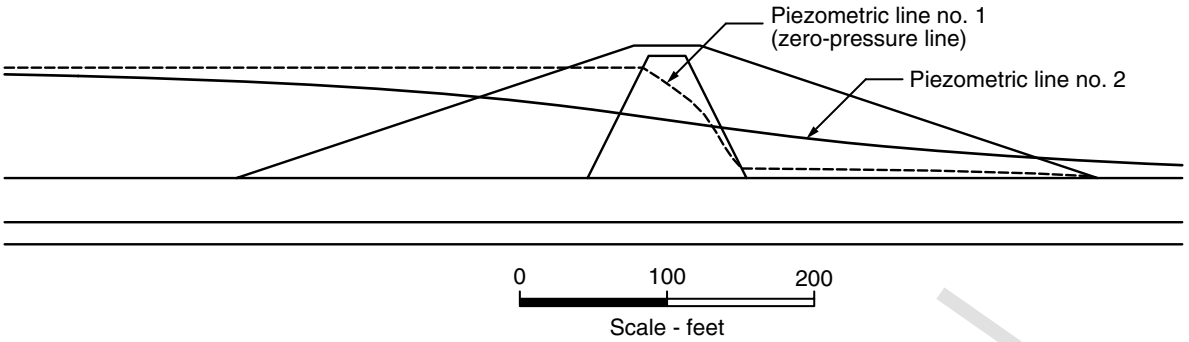



**Figure 6.41** Two piezometric lines used to represent the pore water pressures for an embankment dam.

Results of the stability calculations for the second example can be improved by using more than one piezometric line. To illustrate this, a second piezometric line was established based on the pore water pressures at the bottom of the foundation clay layer (top of sand layer). The second piezometric line is shown in Figure 6.41 and better represents the pore water pressures (total head) in the sand and lower part of the clay. An additional set of slope stability calculations was performed using the second piezometric line to define the pore water pressures in the bottom half of the clay layer, and the original first piezometric line was used to define the pore water pressures in the upper half of the foundation clay as well as in the embankment. Calculations were performed for circles tangent to elevation 197, which is the bottom of the clay layer. The factor of safety computed using the two piezometric lines was 1.36, which is almost identical to the value (1.37) that was obtained when the pore water pressures were obtained by interpolation of the values from the finite element analysis. Thus, the use of multiple—in this case, two—piezometric lines can improve the solution obtained using piezometric lines, although the very close agreement between the results with two piezometric lines and interpolation may be somewhat fortuitous. Also, it is not always as easy to establish appropriate piezometric lines when multiple lines are to be used.

## SUMMARY

When flow is predominately horizontal (vertical equipotential lines), both a phreatic surface and single piezometric line can be used to approximate the pore water pressures relatively well, with an error of only a few percent at most. There appears to be little difference between these two representations of pore water pressure (phreatic surface and piezometric line); the refinement of using a phreatic surface representation rather than a simpler piezometric line does not appear to produce a better representation. In fact, the phreatic surface may actually produce less accurate results than the piezometric line.

For cases where the flow is not predominately horizontal and as the component of flow and head loss in the vertical direction increases, neither a phreatic surface nor a single piezometric line represent pore water pressures well; both may result in errors on the unsafe side. In such cases, care must be exercised in selecting an appropriate representation of pore water pressures. It is probably better to use an appropriate seepage solution and interpolate pore water pressures. Currently, there are several good finite element software programs available for seepage analyses. Use of pore water pressures from a finite element analysis requires some effort to interpolate pore water pressures from nodal point values to points along a slip surface (base of slices), but robust and efficient interpolation schemes are available for this as well.

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